Two interacting fermions in a one-dimensional harmonic trap: Matching the Bethe ansatz and variational approaches

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In this work, combining the Bethe ansatz approach with the variational principle, we calculate the ground-state energy of the relative motion of a system of two fermions with spin up and spin down interacting via a delta-function potential in a one-dimensional (1D) harmonic trap. Our results show good agreement with the analytical solution of the problem, and provide a starting point for the investigation of more complex few-body systems where no exact theoretical solution is available.

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I. INTRODUCTION

Few-body quantum systems composed of atoms and molecules are some of the most simple structures that constitute the building blocks of matter. Despite this simplicity their study has recurrently been challenging. One of the reasons for this being that one cannot make use of standard statistical methods and still have enough degrees of freedom to make it a complex problem, often not solvable for as few as three bodies. The interest in few-body systems is manifold and has over time appeared in nuclear and particle physics as well as in atomic and molecular studies. Currently, a renewed interest has also emerged in relation to the experimental study of Bose and Fermi gases since few-body interactions may play a far from trivial role in their behavior [1]. Moreover, the recent and impressive development of the technology associated with the study of the Bose-Einstein condensation phenomena in fields such as ultracold gases, Mott insulators, and optical lattices led to the possibility of controlling in an increasingly precise way the number of atoms trapped in a well.

In particular, a great deal of interest has been devoted to the study of distinguishable trapped few-fermion systems. The most recent experimental achievement being the realization of a system of two fermionic atoms of ^{6}Li with tunable interactions [2,3]. In this experiment, the ground-state energy of the system was measured and compared to an analytical result that exists in this particular case [4] (see also [5]), which, however, is not extendable if one includes more atoms. Therefore a good approximation that may be generalized to more than two atoms is of interest. It is worthwhile to mention here that the Hamiltonian employed in the calculation of the ground-state energy of this two-fermion experiment is basically equivalent to the one used to discuss the existence of exotic pairing mechanisms closely related to the elusive Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) [6,7] state. In that case, one deals with a higher number of particles, in addition to a spin imbalance, and the thermodynamical Bethe ansatz coupled to a local density approximation was used to discuss the resulting phase diagrams and density profiles of trapped fermionic ^{6}Li atoms in one-dimensional (1D) tubes [8–10].

With the above motivations, viewing the prospect of new few-body experiments [11], we propose an alternative possibility, a variational approach based on the use of the Bethe-ansatz solution for a system with delta-function interactions. Our choice will take into account the knowledge of the exact solution of the 1D many-body system with repulsive or attractive delta-function potentials [12–14] and consider the trapping as a kind of perturbation. By this we mean that the bulk of our ansatz is supposed to grasp the behavior of the interacting particles which happen to be trapped in a harmonic well. This is an unexplored possibility and, for that matter, one that has the potential to be systematically generalized from two to more particles. Our approach consists of calculating the ground state of the few-fermion model keeping in mind the variational principle, such that the actual ground-state energy is smaller than the chosen state of the Hamiltonian with delta interactions, which we know exactly by the Bethe-ansatz methods, plus a part that is the mean value of the harmonic potential for our ansatz.

In the following we develop our systematics for the variational calculation of the ground state of a two-fermion system. The next section will be devoted to set forth the system of two interacting fermions that we are investigating, then in Sec. III we introduce our variational ansatz, which as mentioned is inspired in a paradigmatic solution for 1D systems [12–15], the Bethe-ansatz [16]. In Sec. IV we present our results for the repulsive and attractive cases and in Sec. V these results are compared with the ones obtained by the authors of [4,5] for the relative motion. In the Appendix we provide details concerning the construction of the Bethe ansatz part in absolute coordinates and briefly discuss its extension to the general case of N fermions.

II. SYSTEM

Let us consider a system of two interacting fermions, for instance, two fermionic atoms with mass m in an axially symmetric harmonic trap with angular frequency ω. Such a system can be described by the following Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + V_A(x_1, x_2) + V_I(x_1 - x_2), \tag{2.1}$$

where x_1 and x_2 denote the position of the two fermions and $V_A(x_1, x_2)$ represents the trapping potential

$$V_A(x_1, x_2) = \tfrac{1}{2}m\omega^2 {x_1}^2 + \tfrac{1}{2}m\omega^2 {x_2}^2. \tag{2.2}$$

For sufficiently low energies the interaction potential V_I can be taken as a delta-function contact potential such that

$$V_I(x_1 - x_2) = 2c\delta(x_2 - x_1), \tag{2.3}$$

where c is the interaction strength. The potential is repulsive or attractive, respectively, for $c > 0$ or $c < 0$.

Here, as the harmonic potential and the kinetic energy are quadratic, it is convenient to separate the relative motion from the center-of-mass motion. This can be easily attained by using the center of mass and relative coordinates given by

$$X = \frac{x_1 + x_2}{2}, \quad x = x_2 - x_1. \tag{2.4}$$

One can then decompose the total Hamiltonian in the center of mass $H_{\rm CM}$ and relative motion $H_{\rm rel}$ parts

$$H_{\rm CM} = -\frac{\hbar^2}{2M}\frac{\partial^2}{\partial X^2} + \frac{1}{2}M\omega^2 X^2, \tag{2.5}$$

$$H_{\rm rel} = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + 2c\delta(x) + \frac{1}{2}\mu\omega^2 x^2. \tag{2.6}$$

In the above, $\mu = \frac{m}{2}$ is the reduced mass and $M = 2m$ the total mass. It can be seen that the eigenfunctions and eigenenergies of $H_{\rm CM}$ are those of the harmonic oscillator. Notice that for the general N case, by the use of Jacobi coordinates, the Hamiltonian is also separable (see the Appendix).

Now, for the Hamiltonian $H_{\rm rel}$, we shall apply the variational principle

$$E_{\rm GS} \leqslant \frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}, \tag{2.7}$$

where $\psi(x)$ is a continuous trial wave function.

The novelty in our approach is that the trial function, which we denote $\psi(x, \{\alpha, L\})$, will encompass the Bethe ansatz concept [12–14,16]. The parameter α controls the decay of the trial function outside the trap and L indicates the limit where this decay starts. Inside the trap, where the contact interaction is relevant, the trial function will take the form of the Bethe ansatz. As usual a variation of these parameters provides a minimal value, which should approximate the ground state of the system. In that way we have a wave function that gives a realistic picture of the physical processes involved.

III. ANSATZ

In the present section we exhibit our variational ansatz. Further details concerning the construction of the Bethe ansatz part can be found in the Appendix, where we also briefly discuss its extension.

As shown in Fig. 1, there are three relevant regions for a wave function of our system. We can delineate these regions by the parameter L. Our variational ansatz assumes then the following configuration:

$$\psi = \begin{cases} \psi_I = e^{-\alpha(x+L)^2}\psi_{II}(-L), & -\infty < x < -L, \\ \psi_{II} = (e^{ikL}e^{-ikx} + e^{ikx})\Theta(x) & \\ \qquad + (e^{ikL}e^{ikx} + e^{-ikx})\Theta(-x), & -L < x < +L, \\ \psi_{III} = e^{-\alpha(x-L)^2}\psi_{II}(+L), & +L < x < +\infty, \end{cases} \tag{3.1}$$

where Θ is the Heaviside step function.

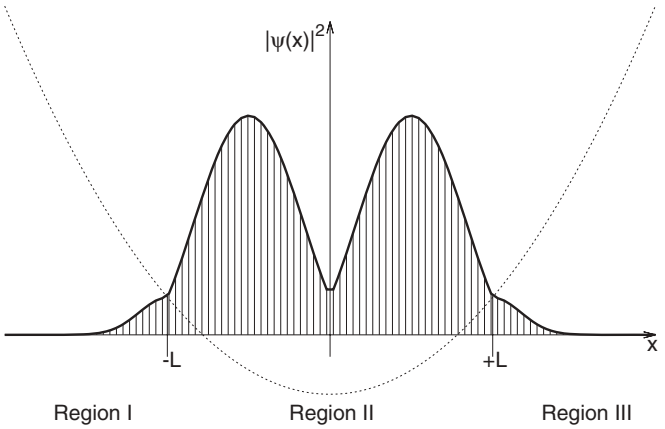

FIG. 1. Schematic representation of the normalized probability density $|\Psi(x)|^2$ in the relative coordinates system in arbitrary scale and Rydberg atomic units. The variational parameter L may be used to delineate three regions according to the boundaries with respect to the harmonic potential.

In region II $(-L < x < +L)$ the two-fermion system is subject to the contact potential and the harmonic trap. Due to the symmetry of the system, in the vicinity of the central axis, where $(x = 0)$, the interaction term is dominant. In other words, any contact interaction takes over the harmonic potential. For this reason we assume that it is possible to approximate the wave function in region II by the wave function that describes a system with two distinct fermions with a contact interaction. Historically, such systems were studied in 1D lattices of size "L" and periodic boundary conditions, being exactly solved in Refs. [13,14]. Later major contributions for this problem where given by Ref. [12] followed by others, such as Refs. [17] and [18–21].

Therefore, our choice for a trial wave function in this region corresponds to the Bethe ansatz solution for fermions interacting through a delta-function potential in relative coordinates, such that ψ_{II} is built as the eigenfunction of the interaction Hamiltonian

$$H_{\rm int}\psi_{II} = \left[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} + 2c\delta(x)\right]\psi_{II} = \frac{\hbar^2}{2\mu}k^2\psi_{II}.$$

More importantly, this means that ψ_{II} will correspond to the eigenfunction of $H_{\rm int}$ with energy $E_{\rm int} = \frac{\hbar^2}{2\mu}k^2$ for all the "quasimomenta" k's that satisfy the following equation:

$$e^{ikL} = \frac{k + i\frac{2\mu}{\hbar^2}c}{k - i\frac{2\mu}{\hbar^2}c}, \tag{3.2}$$

known as the Bethe ansatz equation.

Hence, for each value of the coupling c we need to determine which value of the quasimomentum k satisfies (3.2) for our choice of state, here the ground state to have ψ_{II} completely defined.

A careful analysis of Eq. (3.2) shows that the possible values for the quasimomenta k depend on the sign of c [12]. For the repulsive case $(c > 0)$, only real k's are ground-state solutions of the Bethe ansatz equations (3.2) and, accordingly, are the values entering in ψ_{II}. For the attractive case $(c < 0)$ the k's

composing the ground state are pure imaginary numbers. We will then consider both cases separately.

Our proposal consists then in building the central part of our trial function as the Bethe ansatz wave function for the relative motion of two distinct fermions interacting via a delta function. This problem is completely solvable and in its generality applied to any number of fermions [12–14]. Notice that in the literature one usually considers the contact interaction as a perturbation to the harmonic potential Hamiltonian. We show here how to use the full strength of the Bethe ansatz in a variational approach.

Before we proceed to the analysis of the repulsive and attractive cases we have still to explain how to deal with the continuity of the wave function on the boundaries between the three regions. The continuity condition in all the intervals dictates that

$$\psi_I(-L) = \psi_{II}(-L), \quad \psi_{II}(+L) = \psi_{III}(+L).$$

In regions I and III the harmonic potential is the only one present, so the simplest choice that takes into account the system behavior should be an eigenfunction of the harmonic oscillator Hamiltonian. As we expect a rapid decay of the probability density in these regions, ψ_I and ψ_{III} have the form of a Gaussian and depend on another variational parameter α. It is important to notice that although our choice is continuous for all x, its first derivative is not. Later we also consider the contribution of this discontinuity to the total value of the ground-state energy.

IV. RESULTS

A. Repulsive interaction, ($c > 0$)

In this case just some values of k, the *purely real* ones, satisfy (3.2) for the ground state, therefore in this section we only consider $k \in \mathbb{R}$. To apply the variational principle, we first compute the normalization of the wave function, which yields

$$\langle\psi|\psi\rangle = 2\sqrt{\frac{\pi}{2\alpha}}[1+\cos(kL)] + \frac{4}{k}[kL+\sin(kL)] \tag{4.1}$$

and then the expectation value of $H_{\rm rel}$, the value of which is

$$\begin{aligned}\langle\psi|H_{\rm rel}|\psi\rangle &= \frac{\hbar^2\alpha}{\mu}\sqrt{\frac{\pi}{2\alpha}}[1+\cos(kL)] + \frac{2\hbar^2k^2L}{\mu} \\ &+ \mu\omega^2[1+\cos(kL)]\left[\frac{1}{2}\frac{\sqrt{\pi}}{(2\alpha)^{\frac{3}{2}}} + L^2\sqrt{\frac{\pi}{2\alpha}} + \frac{L}{\alpha}\right] \\ &+ \frac{\mu\omega^2}{3k^3}[2L^3k^3 + 3(k^2L^2-1)\sin(kL) + 3kL\cos(kL)].\end{aligned} \tag{4.2}$$

In the expression (4.2) we also considered the contribution from the discontinuity of the wave-function first derivative at $x = \pm L$, that is,

$$\begin{aligned}&\lim_{\epsilon\to 0}\left[\int_{-L-\epsilon}^{-L+\epsilon}\psi^*(H_{\rm rel}\psi)dx + \int_{L-\epsilon}^{L+\epsilon}\psi^*(H_{\rm rel}\psi)dx\right] \\ &\quad = -\frac{2\hbar^2k}{\mu}\sin(kL),\end{aligned}$$

which comes from the kinetic term of $H_{\rm rel}$.

The Bethe ansatz equations (3.2) for the ground state in the repulsive interaction reduce then to

$$k = \frac{2}{L}\arctan\left(\frac{2\mu c}{\hbar^2 k}\right), \tag{4.3}$$

which are much simpler to solve.

We have then all the necessary ingredients to proceed with the numerical minimization of $\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}$ with respect to the parameters α and L. Basically, to each assigned c we sweep over all values of α and L, calculate k for each L, and establish the parameters α^* and L^* such that $\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}$ takes the least possible value. In this way we determine the ground-state energy of the two-fermion system as a function of the coupling c via the variational principle, where the trial wave function is constructed by means of the Bethe ansatz. This result is depicted in Fig. 2 using the physical variables ϵ and $a_{\rm 1D}$. We give more details in the next section where we also compare this result with the analytical solution [4].

Limiting case: Notice that in the limit $L \to 0, c \to 0$ (harmonic oscillator) the expression (4.2) reduces to

$$\lim_{L\to 0}\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\hbar^2\alpha}{2\mu} + \frac{\mu\omega^2}{8\alpha}. \tag{4.4}$$

Upon extremization of the total energy with respect to α in the limit $L \to 0$, the minimum value is the one for the value α^* of the parameter

$$|\alpha^*| = \frac{\mu\omega}{2\hbar}, \tag{4.5}$$

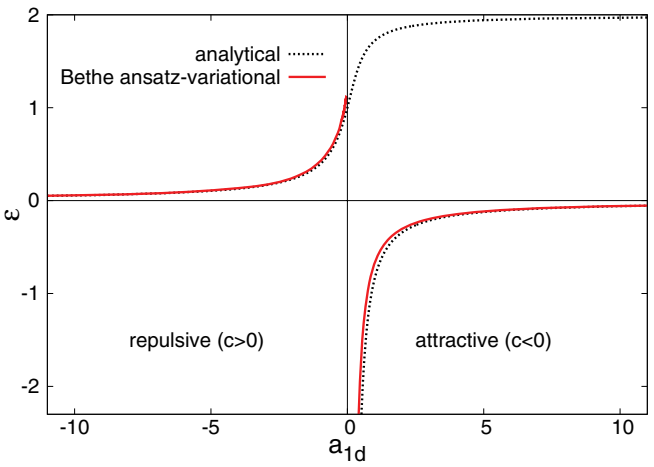

FIG. 2. (Color online) Energies for the ground state of the relative motion for two distinct fermions interacting via a delta-function potential and confined in a harmonic trap of frequency ω. The analytically obtained energy levels (black dotted line) are compared to the results for the energy given by the combination of the Bethe ansatz with the variational principle (red solid line).

such that

$$\lim_{L\to 0}\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}\bigg|_{\alpha=\alpha^*}=\frac{1}{2}\hbar\omega, \tag{4.6}$$

which, as expected, is simply the ground-state energy of the harmonic oscillator.

B. Attractive interaction, ($c < 0$)

In this case only the purely imaginary values of k satisfy the Bethe ansatz equations (3.2) for the ground state; for this reason we will consider $k \in \mathbb{C}$. Thus, it is convenient to define $k = ik'$, $k' \in \mathbb{R}$. In terms of k', we can write (3.2) as

$$e^{-k'L}=\frac{k'+\frac{2\mu}{\hbar^2}c}{k'-\frac{2\mu}{\hbar^2}c}, \tag{4.7}$$

which can be solved by numerical methods.

To apply the variational principle, we first compute the normalization of the wave function, which yields

$$\langle\psi|\psi\rangle=2e^{-k'L}\left\{\sqrt{\frac{\pi}{2\alpha}}[1+\cosh(k'L)]+\frac{2}{k'}[Lk'+\sinh(k'L)]\right\} \tag{4.8}$$

and then the mean value of $H_{\rm rel}$, obtaining

$$\begin{aligned}\langle\psi|H_{\rm rel}|\psi\rangle&=\frac{\hbar^2\alpha}{\mu}\sqrt{\frac{\pi}{2\alpha}}e^{-k'L}[1+\cosh(k'L)]\\&-\frac{2\hbar^2k'^2L}{\mu}e^{-k'L}+\frac{\mu\omega^2}{3k'^3}e^{-k'L}[2L^3k'^3\\&+3(k'^2L^2+1)\sinh(k'L)-3k'L\cosh(k'L)]\\&+\mu\omega^2e^{-k'L}[1+\cosh(k'L)]\left[\frac{1}{2}\frac{\sqrt{\pi}}{(2\alpha)^{\frac{3}{2}}}+L^2\sqrt{\frac{\pi}{2\alpha}}+\frac{L}{\alpha}\right],\end{aligned} \tag{4.9}$$

where again we consider the contribution of the discontinuity of the wave function derivative at the points $x = \pm L$

$$\lim_{\epsilon\to 0}\left[\int_{-L-\epsilon}^{-L+\epsilon}\psi^*(H_{\rm rel}\psi)dx+\int_{L-\epsilon}^{L+\epsilon}\psi^*(H_{\rm rel}\psi)dx\right]=\frac{2\hbar^2k'}{\mu}e^{-k'L}\sinh(kL),$$

into the expression (4.9).

As in the attractive case we numerically minimize the mean energy $\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}$ with respect to the parameters α and L and determine the ground-state energy of the system as a function of the coupling c. This result is depicted in Fig. 2. In the next section we make a comparison with the analytical solution obtained in Ref. [4].

Limiting case: Notice again that in the limit $L \to 0$, $c \to 0$ (harmonic oscillator)

$$\lim_{L\to 0}\frac{\langle\psi|H_{\rm rel}|\psi\rangle}{\langle\psi|\psi\rangle}=\frac{\hbar^2\alpha}{2\mu}+\frac{\mu\omega^2}{8\alpha}. \tag{4.10}$$

Upon extremization of the total energy with respect to α in the limit $L \to 0$, we obtain $|\alpha^*| = \frac{\mu\omega}{2\hbar}$, such that we find again the ground-state energy of the harmonic oscillator.

V. COMPARISON

The results obtained in the previous section by means of the Bethe ansatz and the variational principle to find the two-fermion system ground state as a function of the coupling parameter are presented in Fig. 2 (red solid line). For convenience and to compare with known results we are using the following variables:

$$\epsilon=\frac{E_{GS}-\frac{1}{2}\hbar\omega}{\hbar\omega}\quad a_{\rm 1D}=-\frac{\hbar^{3/2}}{2c}\sqrt{\frac{\omega}{\mu}}, \tag{5.1}$$

where ϵ denotes the energy of the ground state shifted by the zero-point energy in $\hbar\omega$ units and $a_{\rm 1D}$ is the 1D scattering length.

The analytical solution in relative coordinates for a system of two distinct fermions interacting via a delta function and confined in a harmonic trap was first obtained by the authors of Ref. [4]. Basically, in this work they expanded the unknown wave function in a complete set of the simple harmonic oscillator functions. Later, these results were generalized to different geometries of the trapping potential in Ref. [5], and among others in Refs. [22–26].

Essentially, the following implicit equation determining the eigenenergies of the relative motion in a 1D harmonic potential was obtained

$$2a_{\rm 1D}=\frac{\Gamma\left(-\frac{\epsilon}{2}\right)}{\Gamma\left(-\frac{\epsilon}{2}+\frac{1}{2}\right)}, \tag{5.2}$$

where $\Gamma(x)$ is the complete gamma function.

This solution for the ground state is plotted in Fig. 2 (black dotted line). We can observe a very good agreement between this result and the result that we obtain combining the Bethe ansatz and the variational principle (red solid line). This places our ansatz as a potential candidate for the extension to more than two fermions, where an analytical solution does not exist. The fact that the measured properties of this system [2] may, with a good agreement, be compared with the theoretical results [4] makes this subject even more captivating.

VI. CONCLUSION

In this work we obtained the ground-state energy of two distinct fermions in a 1D harmonic trap within a variational approach, but from a distinct perspective, aiming a new view for the problem of few fermions. The reasoning beneath our variational ansatz choice was to exploit the exact solution for the 1D system of fermions interacting by means of a contact potential solution, the Bethe ansatz. Usually, in the literature, one takes a route different from ours by considering the harmonic trap Hamiltonian as the relevant one and the contact interaction as a perturbation. But, since for delta-function interactions, we have at our disposal the Bethe ansatz technology it is almost natural to use it. Thus, we chose a trial wave function for this system that contains a great deal of information about the physics of the two fermions inside the trap and supplement it by the knowledge of the harmonic

oscillator Hamiltonian. The good agreement between our results and existing analytical results shows that our ansatz fulfills our expectation and has the potential to shed light on the spectrum of strongly correlated few-body quantum systems. Using the methods established in this work it is, in principle, possible to extend our studies to more complex systems, composed of three or more fermions, which are currently of experimental interest [11], and also in this case one can profit of the exact solution for the contact interaction. The procedure for higher N brings, however, a substantial operational growth as there are $N! \times N!$ coefficients of the Bethe ansatz to be determined and the number of regions of the complete variational ansatz, as in Eq. (3.1), increases correspondingly.

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APPENDIX

We develop here, in detail, how we built the trial wave function for the two-body problem and then we indicate how to apply the same principles for a higher number of fermions.

The rationale we use in our construction is the Bethe ansatz method for obtaining the energy spectra of exactly solvable Hamiltonians. Let us then consider two fermions interacting through a delta-function potential in a 1D system with periodicity L, which has the following Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + 2c\delta(x_1 - x_2), \tag{A1}$$

where x_1 and x_2 are the position of each fermion and c is the interaction strength. The most general wave function for such a system in absolute coordinates in the region $x_1, x_2 \in [-L/2, +L/2]$ can be written as Ref. [12]

$$\begin{aligned}\psi(x_1,x_2) = &\left[a_{12}^{12}e^{i(k_1x_1+k_2x_2)} + a_{21}^{12}e^{i(k_2x_1+k_1x_2)}\right]\Theta(x_2 - x_1)\\ &+ \left[a_{12}^{21}e^{i(k_1x_2+k_2x_1)} + a_{21}^{21}e^{i(k_2x_2+k_1x_1)}\right]\Theta(x_1 - x_2),\end{aligned} \tag{A2}$$

where k_1 and k_2 are the "quasimomenta" for the fermions and the coefficients "a" are to be determined by physical arguments. The action of the Hamiltonian on the wave function results in

$$H\psi(x_1,x_2) = E\psi(x_1,x_2) + \text{undesirable terms}, \tag{A3}$$

where the "undesirable terms" are functions of k_1 and k_2. When, as usual, one requires that these terms be null, k_1 and k_2 must satisfy certain consistency relations known as the Bethe ansatz equations [12]. These depend on the wave-function symmetry. Energy and momentum are given, respectively, by

$$E = k_1^2 + k_2^2, \quad K = k_1 + k_2. \tag{A4}$$

Once the system is in a spin singlet configuration (antisymmetric) the wave function must be spatially symmetric

$$\psi(x_2,x_1) = \psi(x_1,x_2), \tag{A5}$$

this implies that

$$\begin{aligned}a_{12}^{12} &= a_{12}^{21} \equiv a_{12},\\ a_{21}^{12} &= a_{21}^{21} \equiv a_{21}.\end{aligned} \tag{A6}$$

Besides, the periodic boundary conditions

$$\psi(x_j = -L/2) = \psi(x_j = +L/2), \quad j = 1,2 \tag{A7}$$

lead to the relations

$$\begin{aligned}a_{12} &= a_{21}e^{ik_1L},\\ a_{21} &= a_{12}e^{ik_2L}.\end{aligned} \tag{A8}$$

It can be shown that, for the ground state, $K = 0 \Rightarrow k_2 = -k_1$. Therefore, we can write the symmetric wave function with the periodic boundary conditions in terms of the absolute coordinates as

$$\begin{aligned}\psi(x_1,x_2) = &\, a_{21}[e^{ik_1L}e^{ik_1(x_1-x_2)} + e^{-ik_1(x_1-x_2)}]\Theta(x_2 - x_1)\\ &+ a_{21}[e^{ik_1L}e^{-ik_1(x_1-x_2)} + e^{ik_1(x_1-x_2)}]\Theta(x_1 - x_2).\end{aligned} \tag{A9}$$

Then, for convenience, defining $x = x_1 - x_2$ and $k \equiv k_1$, we obtain our ansatz in the relative coordinates system

$$\begin{aligned}\psi(x) = &\, a_{21}[e^{ikL}e^{ikx} + e^{-ikx}]\Theta(-x)\\ &+ a_{21}[e^{ikL}e^{-ikx} + e^{ikx}]\Theta(x).\end{aligned} \tag{A10}$$

The constant a_{21} is obtained by the normalization condition. The above wave function is the eigenfunction of the interaction Hamiltonian in relative coordinates in the $-L \leqslant x \leqslant L$ interval, and constitutes the central part of our ansatz. In the $x \to \pm\infty$ limits it is expected that the wave function exhibits an asymptotic behavior similar to the behavior of an harmonic oscillator wave function with exponential decay. Then the trial wave function in the other intervals is obtained by the condition of continuity of the wave function on the boundaries between all regions.

The generic N fermion case can be dealt with in an analogous way. The wave function in absolute coordinates is similar to that of Eq. (A2) with coefficients $a_{\ldots\lambda\mu\ldots}^{\ldots lm\ldots}$ (the indices run from 1 to N). It is possible then to proceed in the same way as before for all other terms. Requiring the same physical principles as above for all the $N!$ regions of the type $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_N \in [0,L]$ respecting the Bethe ansatz with periodicity L, a complete wave function in absolute coordinates can be constructed. For instance, for the case $N = 3$ we have, in a compact form

$$\begin{aligned}\psi(x_1,x_2,x_3) = &\sum_{\substack{l,m,n=1\\ l\neq m\neq n}}^{3}\sum_{\substack{\lambda,\mu,\nu=1\\ \lambda\neq\mu\neq\nu}}^{3} a_{\lambda\mu\nu}^{lmn}e^{i(k_\lambda x_l + k_\mu x_m + k_\nu x_n)}\\ &\Theta(x_n - x_m)\Theta(x_m - x_l).\end{aligned}$$

An important element here is that the coefficients of the Bethe ansatz are related through a well-established transformation where the operators are given by Refs. [12,27] and satisfy the Yang-Baxter equation [13,15]. It is important to note that, in this case, it is convenient to change from the absolute

coordinates system to Jacobian coordinates and it is possible to show that for both the contact and trapping interactions the resulting Hamiltonian is separable in the center of mass and relative coordinates [28]. This enable us to obtain, knowing the Bethe ansatz for absolute coordinates, the result for the relative coordinates Hamiltonian. In other words, the described procedure, when N is increased, though cumbersome (the number of coefficients increases as $N!^2$), allows one to build the Bethe ansatz part of the whole variational ansatz. The caveat, of course, is that the number of regions, such as in Eq. (3.1) where one has to use continuity conditions, also increases accordingly.

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