COMPACT ε - CONVEX HYPERSURFACES

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- Trabalho de Pesquisa -

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Compact ϵ-Convex Hypersurfaces

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1. Introduction. A classical result due to Hadamard ([1]) and its generalization, due to Hopf ([2]), prove that if a compact hypersurface of the Euclidean space has Gauss-Kronecker curvature everywhere different from zero then the hypersurface a) is diffeomorphic to a sphere, b) is embedded, and c) is the boundary of a convex body. The hypothesis on the Gauss-Kronecker curvature and the compactness is equivalent (as a hypersurface of the Euclidean space) to the hypersurface having the principal curvatures with the same sign and with absolute value greater than a positive constant. This last statement was used by Eschenburg in [3] to introduce the concept of a ϵ-convex hypersurface in an arbitrary Riemannian manifold, namely, a hypersurface whose principal curvatures have the same sign and absolute value greater than ϵ.

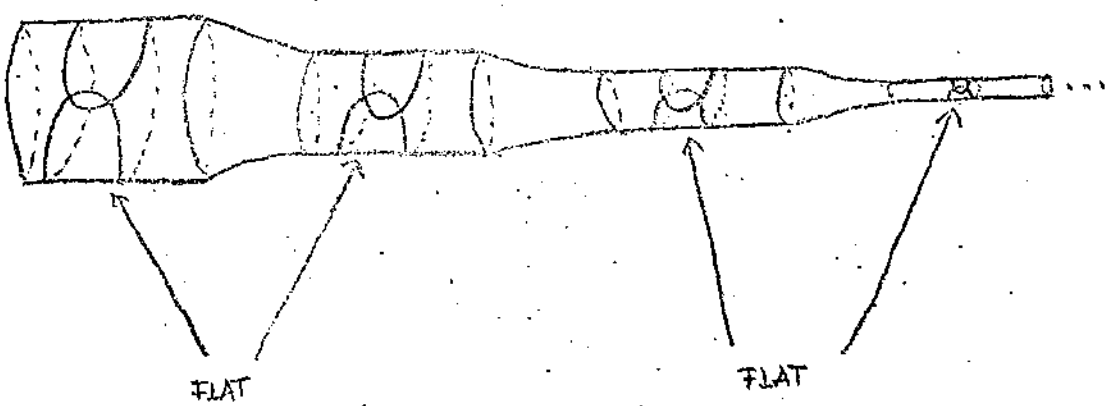
Eschenburg asks if it is possible to characterize all compact ϵ-convex hypersurfaces of a Riemannian manifold. Being more specific, we can ask under which conditions we can conclude that a given compact ϵ-convex hypersurface of a complete Riemannian manifold has to satisfy a), b) and c) as stated above. For the case that the ambient space has non negative curvature, Eschenburg proved that condition a) is always true. Precisely, he proved that a compact ϵ-convex hypersurface of a complete Riemannian manifold with non negative sectional curvature, for any $\epsilon > 0$, is the boundary of an immersed disk in the space ([3]). A complete answer for the above question (in the sense that it answers conditions b) and c)), but less general than the Eschenburg's Theorem, is a result of do Carmo and Warner which extends the results of Hadamard and Hopf to any simply connected space of constant curvature ([4]). Another important theorem due to Tribuzy ([5]) answers also affirmatively conditions a), b) and c) when the ambient space is a complete non compact Riemannian manifold whose sectional curvatures are positive and bounded.

Our main aim in this paper is to prove the following theorem:

Theorem A. *Let N be a complete Riemannian manifold. Then there exists $L = L(N) \geq 0$, determined in terms of geometric invariants of N, such that if M is a compact ϵ-convex hypersurface of N with $\epsilon > L$ then M is diffeomorphic to a sphere, embedded, and is the boundary of a convex body.*

The value of $L(N)$ as determined in the proof of Theorem A is not generally the best one, that is, the minimum non negative real number for each conditions a), b) and c) are true. If N is compact, then it follows from the proof of Theorem A that $L(N)$ is finite. It is also finite if N is a homogeneous manifold (see remarks i) and ii) of §3). For the special case that N is a simply connected space of constant curvature, we can adapt the proof of Theorem A, using the techniques introduced in [3], to prove it with $L = 0$ (we reobtain this way do Carmo, Warner's Theorem, although in a weaker form, compared with its original statement, see [4]).

We observe that if N is just complete then $L(N)$ can be infinite, as shows the picture below. The circles drawed in the picture also show that the theorem is actually false for such an N. In this case this means that for any $\delta > 0$ there exits an ϵ-convex hypersurface of N with $\epsilon > \delta$ which is not embedded:



If we consider just condition a), then Eschenburg's result implies that we can take $L = 0$ if N has non-negative curvature. We also prove here that if we consider just condition a), then we can take $L = 0$ in many Riemannian homogeneous manifolds, for example, a Lie group with a left invariant metric, or a symmetric space whose irreducible factors are not of compact type (section §5). Since most of these spaces have negative curvature, this result is not contained in Eschenburg's Theorem. We also observe that the boundary of a tubular neighbourhood of a closed geodesic in a space of negative curvature shows that such a result is not generally true in these spaces.

We think that it would be interesting to compute the best value of $L(N)$ (or at least to obtain an estimate), in terms of geometric invariants of N, for which just condition a) is true and for which conditions a), b) and c) are true.

The proof of Theorem A, apart from some technical definitions, is very simple, going back to the classical proof of Hadamard by defining, with the aid of a "referencial frame" defined on the hypersurface, a map from the hypersurface to the sphere of the same dimension (this map generalises the usual Gauss map of a hypersurface in the Euclidean space) and proving that this map, under the hypothesis of the theorem, is a diffeomorphism. As in the Euclidean space, this fact will also imply that the hypersurface is embedded and boundaries a convex body. This technique is an improvement of the one introduced by the author in [6] for studying hypersurfaces of a Lie group. In fact, Theorem 9 of [6] is a corollary of the results of this paper (see section 5).

2. Proof of the Theorem A.

Let us assume $dim(N) = n + 1$.

Given a geodesic $\gamma : \mathbf{R} \to N$, denote by π_γ the projection over γ, namely:

$$\pi_\gamma(p) = q \in \gamma(\mathbf{R}) \longleftrightarrow d(p, \gamma(\mathbf{R})) = d(p, q), \quad p \in N$$

where d is the Riemannian distance in N.

Given γ, let U_γ be the biggest open subset of N such that π_γ is a well defined differentiable map, without critical points in U_γ. For a given γ, we define a function $d_\gamma : U_\gamma \to \mathbf{R}$ by $d_\gamma(p) := d(\pi_\gamma(p), \gamma(0))$.

Let $D := \{X_\gamma := \text{grad}(d_\gamma) \mid \gamma \text{ is a geodesic of } N\}$. Given $X \in D$, the domain of X is $U_X := U_\gamma$, where γ is such that $X = \text{grad}(d_\gamma)$.

Let $\delta(N) := \sup\{| < \nabla_v X, v >_p | \mid X \in D,\ p \in U_X,\ v \in T_p(N),\ ||v|| = 1\}$, where ∇ is the Levi-Civita connection of N.

Choose any $p \in N$, denote by $B_p(r)$ the geodesic ball of N centered at p with radius $r > 0$, and set:

a) $D_p := \{X \in D \mid X = \text{grad}(d_\gamma)$ where γ is a geodesic such that $\gamma(0) = p\}$,

b) For a given $X \in D_p$, set $R(p, X) := \sup\{r \in \mathbf{R}^+ \mid$ there exist $X_1, ..., X_n \in D_p$ such that $B_p(r) \subset U_{X_j}$, $1 \leq j \leq n$, $B_p(r) \subset U_X$, and $X(q), X_1(q), ..., X_n(q)$ are linearly independent, for any $q \in B_p(r)\}$.

Given $p \in N$, let $\bar{R}(p) > 0$ be such that $R(q, X) \geq 2\bar{R}(p)$, for any $q \in B_p(\bar{R}(p))$ and for any $X \in D_q$.

c) $R(N) := \inf_{p \in N} \bar{R}(p)$.

Given $X \in D_p$, we define

d) $G(p, X) := \{\Gamma := \{X, X_1, ..., X_n\} \mid X, X_j \in D_p,\ 1 \leq j \leq n$, and $X(q), X_1(q), ..., X_n(q)$ are linearly independent, for all $q \in B_p(\bar{R}(p))\}$,

and for $X \in D_p$ and $\Gamma \in G(p, X)$, $\Gamma = \{X, X_1, ..., X_{n+1}\}$, $X_1 := X$, we associated the bundle:

e) $E_\Gamma := \{(q, \sum_{j=1}^{n+1} a_j X_j(q)) \mid \sum_{j=1}^{n+1} a_j^2 = 1\}$.

Given $q \in B_p(\bar{R}(p))$, the fiber $E_\Gamma(q)$ of E_Γ, namely

$$E_\Gamma(q) = \{\sum_{j=1}^{n+1} a_j X_j(q) \mid \sum_{j=1}^{n+1} a_j^2 = 1\}$$

is an ellipsoid in $T_q(N)$.

f) A "referencial frame" $\Gamma \in G(p, X)$ such that $\inf\{||v|| \mid v \in E_\Gamma(q), \text{ for any } q \in B_p(\bar{R}(p))\} \geq \inf\{||v|| \mid v \in E_\kappa(q),$ for any $\kappa \in G(p, X)$, for any $q \in B_p(\bar{R}(p))$ $\}$, we will denote by $\Gamma_{p,X}$.

Set

g) $S(p) := \inf\{||v|| \mid v \in E_{\Gamma_{p,X}},\ q \in B_p(\bar{R}(p)), \text{ for some } \Gamma_{p,X} \in G(p, X)\}$,

h) $S(N) := \inf_{p \in N} S(p)$, and finally,

i) $L(N) := \max\{\sqrt{n+1}\,\delta(N)/S(N),\ 2\pi/R(N)\}$.

Let M be a compact ϵ-convex hypersurface of N with $\epsilon > L(N)$. Let us prove that M is diffeomorphic to a sphere, embedded, and boundaries a convex body.

From Bonnet-Meyers Theorem, given any $p \in M$ we have $M \subset B := B_p(R(N))$. Choose $p \in M$, a vector field $X \in D_p$ and a referencial frame $\Gamma \in G(p, X)$. Assume that $\Gamma = \{X_1, ..., X_{n+1}\}$, $X_1 := X$.

Since M is ϵ-convex, we can choose an unitary normal vector field η to M such that $< \nabla_v \eta, v >_q \geq \epsilon$, for any $q \in M$ and for any $v \in T_q(M)$, $||v|| = 1$.

There exists a positive function $f : M \to \mathbb{R}^+$ such that $f(q)\eta(q) \in E_\Gamma(q)$, for any $q \in M$. We define then a map

$$\gamma : M \to \mathbb{S}^n$$

by putting

$$\gamma(q) = (a_1(q), ..., a_{n+1}(q))$$

if

$$f(q)\eta(q) = a_1(q)X_1(q) + ... + a_{n+1}(q)X_{n+1}(q).$$

Clearly, γ is a well defined differentiable map.

Let us prove first that γ is a diffeomorphism. By contradiction, assume the opposite. Then there exist $q_0 \in M$ and $v \in T_{q_0}(M)$, $||v|| = 1$, such that $d\gamma(q_0)(v) = 0$. Since

$$d\gamma(q)(w) = (da_1(q)(w), ..., da_{n+1}(q)(w))$$

we obtain

$$da_1(q_0)(v) = ... = da_{n+1}(q_0)(v) = 0.$$

By another hand, from the equality

$$f(q)\eta(q)) = \sum_{j=1}^{n+1} a_j(q)X_j(q)$$

we obtain

$$df(q)(w)\eta(q) + f(q)\nabla_w\eta(q) = \sum_{j=1}^{n+1} da_j(q)(w)X_j(q) + \sum_{j=1}^{n+1} a_j(q)\nabla_w X_j(q)$$

so that, at $q = q_0$ and at $v \in T_{q_o}(N)$, we obtain

$$df(q_0)(v)\eta(q_0) + f(q_0)\nabla_v\eta(q_0) = \sum_{j=1}^{n+1} a_j(q_0)\nabla_v X_j(q_0).$$

Taking the inner product with v in both sides of this last equality, we get

$$(*) \quad f(q_0) < \nabla_v\eta, v >_{q_0} = \sum_{j=1}^{n+1} a_j(q_0) < \nabla_v X_j, v >_{q_0}$$

From the ϵ-convexity of M, and from the choice of ϵ we obtain

$$f(q_0)\epsilon \leq \sum_{j=1}^{n+1} |a_j(q_0)|| < \nabla_v X_j, v >_{q_0} |$$

$$\leq \delta(N) \sum_{j=1}^{n+1} |a_j(q_0)| \leq \sqrt{n+1}\delta(N).$$

But we have

$$f(q_0) = ||f(q_0)\eta(q_0)||$$

and since

$$f(q_0)\eta(q_0) \in E_\Gamma(q_0)$$

we obtain

$$f(q_0)\epsilon \geq S(N)\epsilon$$

so that

$$S(N)\epsilon \leq \sqrt{n+1}\delta(N)$$

that is

$$\epsilon \leq \sqrt{n+1}\delta(N)/S(N)$$

contradiction!

We prove now that M is embedded. By contradiction, assume the opposite. Since M is compact, it has a self intersection point, say p. Let us choose $X_1, X_2 \in D_p$ such that X_j is normal to M at p, $1 \leq j \leq 2$. Let us consider the "Gauss map" $\gamma : M \to \mathbf{S}^n$ determined by a frame Γ_{p,X_1}. As above, we have that γ is a diffeomorphism. As an immersed hypersurface, p is in the image of two points p_1, p_2 of M and we may assume that X_1 is the normal at p_1. We can not have $X_1 = X_2$ otherwise $\gamma(p_1) = \gamma(p_2) = (1, 0, ..., 0)$, contradiction! Now, let us suppose that X_1 is the gradient vector field of d_γ, where γ is a geodesic of N with $\gamma(0) = p$. Then, p_1 is a critical point of d_γ restricted to M. If $X_1 = -X_2$, then p_2 is also a critical point of d_γ restricted to M. Since p_1, p_2 have the same image p in N, they can not be simultaneously maximum and minimal critical points of d_γ. Therefore, by the compactness of M, d_γ restricted to M has to assume another critical point (maximum or minimum), different from p_1, p_2, contradicting the injectivity of γ (in these points we would have $\gamma = \pm(1, 0, ..., 0)$).

If $X_1 \neq \pm X_2$, then p_1 can neither be a point of maximun nor a minimum of d_γ, and this again, by the compactness of M, will contradicts the injectivity of γ.

Let us prove now that M is convex. Let p_1 and p_2 be points in the same connected component of N/M, and let γ be a minimizing geodesic joining p_1 to p_2. By contradiction, let us assume that γ intersects transversaly M in a point p. Let $X \in D_p$ be such that $X(p)$ is normal to M at p, and suppose that X =grad$(d_{\bar{\gamma}})$. Then p is a critical point of $d_{\bar{\gamma}}$ restricted to M which, since the angle between γ and $\bar{\gamma}$ at p is less then $\pi/2$, is neither a point of global maximun nor a point of global minimun. Therefore, since M is compact, ϕ_{γ} will have at least three different critical points. But this contradicts the injectivety of the "Gauss map" associated to any referencial frame containing X.

3. Remarks.

i) If N is compact, since $\bar{R}(p) > 0$ for any $p \in N$ we have that $R(N) = \inf_{p \in N} \bar{R}(p) > 0$. For the same reason, since $S(p) > 0$ for any $p \in N$, we obtain $S(N) > 0$ and therefore, from i) §2, we obtain $L(N) < \infty$.

ii) If N is a homogeneous manifold then $R(N) = \bar{R}(p) > 0$ and $S(N) = S(p) > 0$, for any $p \in N$ and this implies that $L(N) < \infty$.

4. The case of simply connected spaces of constant curvature.

In these cases, the level sets of d_{γ} are totally geodesic hypersurfaces of N. Therefore, $\delta(N) = 0$. When N is the Euclidean space or the hyperbolic space, then clearly $R(N) = \infty$ so that $L(N) = 0$ in these cases, according to definitions a),...,i) of §2. When N is the sphere, we have $R(N) < \infty$, so that $L(N) > 0$. However, we can prove the theorem for $L = 0$ in the following way.

According to [3], we can deform the hypersurface along its mean curvature vector preserving its ϵ-convexity. We do this deformation until the hypersurface is completly contained in a hemisphere of the sphere. Then we apply Theorem A and we conclude that this (deformed) hypersurface is embedded and boundaries a convex body of the sphere. Now we start coming back through the same deformation and we continue with this deformation until the hypersurface touches the boundary of the hemisphere for the first time. If this first time does not exist it is because the hypersurface was already contained in the hemisphere and we are done. Otherwise, from the ϵ-convexity of the hypersurface we conclude that the hypersurface is itself the boundary of the hemisphere, that is, a totally geodesic hypersurface.

5. Homogeneous spaces

We use here a variation of the proof of Theorem A to prove that condition a) is always satisfied for ϵ-convex compact hypersurfaces in a class of homogeneous spaces.

For, let us consider a ϵ-convex compact hypersurface M of a Riemannian manifold N. Let us assume that there exists $n+1$ vector fields $X_1, ..., X_{n+1}$ of N which are linearly independents over M, $n+1 = dim(N)$. We assume also that these vector fields satisfy the equation

$$(**) \quad < \nabla_u X_j, v >_p + < \nabla_v X_j, u >_p = 0$$

for any $p \in M$ and for any $u, v \in T_p(M)$.

As in the proof of Theorem A, we can define, with the aid of these vector fields, a map $\gamma : M \to S^n$. We claim that this map is a diffeomorphism. Otherwise, we would have:

$$f(q_0) < \nabla_v \eta, v >_{q_0} = \sum_{j=1}^{n+1} a_j(q_0) < \nabla_v X_j, v >_{q_0}$$

for some $q_0 \in M$ and for some $v \in T_{q_0}(M)$, $v \neq 0$, as in the equation (*) in the proof of Theorem A. But it follows from equation (**) that

$$< \nabla_v X_j, v >_{q_0} = 0, \quad j = 1, ..., n+1$$

which contracdicts the ϵ-convexity of M.

It is known that a Killing field satisfies equation (**). Therefore, since a homogeneous spaces has locally as much linearly independents Killing fields as its dimension, it follows that this result applies to these spaces. In special for those homogeneous spaces addmitting globally linearly independents Killing fields (in a number equal to the dimension of the space), as a Lie group with a left invariant metric (in which Killing fields are right invariant vector fields), or a symmetric space whose irreducible factors are not of compact type (taking the transvections along linearly independents geodesics).

Série A: Trabalho de Pesquisa.

01. Marcos Sebastiani - Transformation des Singularités - MAR/89.

02. Jaime Bruck Ripoll - On a Theorem of R. Langevin About Curvature and Complex Singularities - MAR/89.

03. Eduardo Cisneros, Miguel Ferrero e Maria Inés Gonzáles - Prime Ideals of Skew Polynomial Rings and Skew Laurent Polynomial Rings - ABR/89.

04. Oclide José Dotto - ϵ - Dilations - JUN/89

05. Jaime Bruck Ripoll - A Characterization of Helicoids - JUN/89.

06. Mark Thompson, V.B. Moscatelli - Asymptotic Distribution of Liusternik-Schnirelman Eigenvalues for Elliptic Nonlinear Operators - JUL/89

07. Mark Thompson - The Formula of Weyl for Regions With a Self-Similar Fractal Boundary - JUL/89

08. Jaime Bruck Ripoll - A Note On Compact Surfaces With Non Zero Constant Mean Curvature - OUT/89

09. Jaime Bruck Ripoll - Compact ϵ - Convex Hypersurfaces - NOV/89