UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
ESCOLA DE ADMINISTRAÇÃO
PROGRAMA DE PÓS-GRADUAÇÃO EM ADMINISTRÇÃO

Marlon Ruoso Moresco

Set-induced deviations

Porto Alegre

2019

Marlon Ruoso Moresco

Set-induced deviations

Dissertation submitted to the Post-graduation Program in Management at Federal University of Rio Grande do Sul in fulfillment of the requirements for degree of Master of Management.
Supervisor: Marcelo Brutti Righi, PhD

Porto Alegre

2019

CIP - Catalogação na Publicação

Moresco, Marlon Ruoso
Set-induced deviations / Marlon Ruoso Moresco. -- 2019.
29 f.
Orientador: Marcelo Brutti Righi.

Dissertação (Mestrado) -- Universidade Federal do Rio Grande do Sul, Escola de Administração, Programa de Pós-Graduação em Administração, Porto Alegre, BR-RS, 2019.

1. Set-induced deviations . 2. Deviation measures. 3. Acceptance set. 4. Risk management. I. Righi, Marcelo Brutti, orient. II. Título.

Elaborada pelo Sistema de Geração Automática de Ficha Catalográfica da UFRGS com os dados fornecidos pelo(a) autor(a).

Marlon Ruoso Moresco

Set-induced deviations

Dissertation submitted to the Post-graduation Program in Management at Federal University of Rio Grande do Sul in fulfillment of the requirements for degree of Master of Management.

Approved research. Porto Alegre:

Marcelo Brutti Righi, PhD
Dissertation Supervisor

Marcelo Savino Portugal, PhD
Committee Member

Eduardo Horta, PhD
Committee Member

Pablo Cristini Guedes, PhD
Committee Member

Porto Alegre
2019

Acknowledgements

I thank my parents and sister for unconditional support. My girlfriend Angélica for the affection and companionship. And my aunt Lorena for letting me stay at her house for some time.

I am also grateful to my supervisor, which made possible for me to write this work.

Lastly, I thank CAPES for the financial support.

Resumo

Propomos uma classe de medidas de desvio induzidas por um conjunto. Os desvios induzidos pelo conjunto representam a quantidade mínima que uma posição deve encolher para se tornar aceitável. Apresentamos resultados que comprovam sua continuidade e propriedades teóricas financeiras. Mostramos que as propriedades do conjunto determinam as propriedades de nossa classe. Quando o conjunto é radialmente ligado a não constantes, fechado para adição escalar e convexo, os desvios induzidos pelo conjunto são medidas de desvio generalizadas. Estendemos nossa abordagem para o caso em que o conjunto é um conjunto de aceitação com a forma de um conjunto de sub-nível de medidas de desvio. Fornecemos resultados que mostram que quando os desvios induzidos pelo conjunto são provenientes de um conjunto de aceitação de uma medida de desvio generalizado, nossa classe é uma versão em escala da medida original. Também investigamos como as operações no conjunto afetam o desvio induzido pelo conjunto e como as operações de medidas de desvio refletem em seu conjunto de aceitação.

Palavras-chaves: Desvios induzidos por conjuntos; Medidas de desvio; Conjunto de aceitação; Gerenciamento de riscos.

Abstract

We propose a class of deviation measures induced by a set. The set-induced deviations represent the minimum amount that a position must shrink to become acceptable. We present results that prove their continuity and financial theoretical properties. We show that the properties of the set determine the properties of our class. When the set is a radially bounded at non constants, closed for scalar addition and convex the set-induced deviations are generalized deviation measures. We extend our approach for the case where the set is an acceptance set with the form of a sub-level set of deviation measures. We provide results that show that when the set-induced deviations come from an acceptance set of a generalized deviation measure, our class is a scaled version of the original measure. We also investigate how operations on the set affect the set-induced deviation and how operations of deviation measures reflect in its acceptance set.

Keywords: Set-induced deviations; Deviation measures; Acceptance set; Risk management.

Contents

1 Introduction

Since the seminal work of Artzner et al. (1999), the financial risk has been investigated from the theoretical and axiomatic viewpoint. At first, this literature focused on monetary or losses measures. Later, Rockafellar et al. (2006a) introduced the axiomatic approach of generalized deviation measures, as generalizations of the standard deviation and similar measures. These measures have been proved useful in engineering and financial problems, as well other areas. See Rockafellar et al. (2006b), Pflug (2006), Rockafellar et al. (2007), Grechuk et al. (2009) and Krokhmal et al. (2013). Due to the importance of variability in risk measurement, Righi and Ceretta (2016), Furman et al. (2017), Berkhouch et al. (2018), and Righi (2018), for instance, propose risk measures (ρ) that including deviation terms. According to the results of Righi and Borenstein (2018), loss-deviation compositions considered in their study have better performance for optimal portfolio strategies in comparison with their counterpart without deviation term.

Traditionally, deviation measures ($\mathscr{D}$) are derived of some, not necessarily symmetric, distance function (d) from the financial random variable ($X \geq 0$ is a gain, $X \leq 0$ is a loss) to its expectation, i.e., $\mathscr{D}(X) = d(X, \mathbb{E}[X])$, or norm, i.e. $\mathscr{D}(X) = ||X - \mathbb{E}[X]||$. The use of deviations measures obtained by distance functions gained notoriety in the financial literature to quantify the risk after the pioneering study of Markowitz (1952) concerning portfolio theory. We refer as the main examples of these measures the variance (σ^2), standard deviation (σ) and standard lower and upper semi-deviations (σ_- and σ_+, respectively). Another possibility, discussed by Rockafellar et al. (2006a), is the one-to-one correspondence between lower range dominated generalized deviation measures and strictly expectation bounded coherent risk measures. We point out as a classic example of such measures the expected shortfall deviation (ESD) proposed by Rockafellar et al. (2006a) as a deviation measure analog to expected shortfall (ES) (see Acerbi and Tasche (2002)).

Inspired by the acceptance set of risk measures and their associated risk measures, we propose a new alternative to derive deviation measures. We formalize a class of deviation measures induced by a set $\mathscr{A}$. We name this class as set-induced deviations, $\mathscr{D}_{\mathscr{A}}$. We impose no restrictions on the $\mathscr{A}$ that induces $\mathscr{D}_{\mathscr{A}}$. Thereby, given a set, the set-induced deviations are obtained in the following way $\mathscr{D}_{\mathscr{A}}(X) = \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}$. Their value represent the minimum amount necessary to shrink a position to fit the set. We assume that it is possible to invest the excess capital resulting from shrinkage in a risk-free asset r. Note that owing to translation insensitivity, the allocation of part of the position in r does not change deviation of the position, i.e., $\mathscr{D}(\gamma X + (1-\gamma)r) = \mathscr{D}(\gamma X)$, where $\gamma \geq 0$ is the amount to be shirked. We prove continuity and financial theoretical properties of set-induced deviations. We demonstrate that the properties of the set determine the properties of our class of deviation measures. When the set is radially bounded at non constants, closed for scalar addition and convex, $\mathscr{D}_{\mathscr{A}}$ is a generalized deviation measure and admits dual representation. We also demonstrate how operations on the set affect the set-induced deviation.

As our second main contribution, we extend our approach to the case where the set is an acceptance set that has the form of a sub-level set, i.e., $\mathscr{A}_{\mathscr{D}}^{k} = \{X : \mathscr{D}(X) \leq k\}$, where $k > 0$ is a coefficient of aversion to deviation. Higher values of k indicate higher levels of tolerance deviation of position. We name $\mathscr{A}_{\mathscr{D}}^{k}$ as acceptance set induced by deviation measures. We show that when the set-induced deviation comes from an acceptance set of a positive homogeneous deviation measure, it is a scaled version of the original measure, i.e., $\mathscr{D}_{\mathscr{A}_{\mathscr{D}}^{k}}(X) = \frac{\mathscr{D}(X)}{k} = \inf\left\{m \in \mathbb{R}_{+} : \frac{X}{m} \in \mathscr{A}_{\mathscr{D}}^{k}\right\}$. We identify that $\mathscr{D}_{\mathscr{A}_{\mathscr{D}}^{k}}$ inherits the properties of the original deviation measure. We provide examples of acceptance set induced by well-known deviation measures, including variance, standard deviation, and semi-deviations. We investigate how operations with deviation measures reflect in its acceptance set.

We contribute to existing literature because, to the best of our knowledge, there are no similar results in the literature to those presented by us. Previous studies, such as Rockafellar et al. (2006a) and Rockafellar and Uryasev (2013), consider the deviation measures to quantify nonconstancy of a random variable. Our proposal brings a new interpretation to deviation measures. Unlike previous approaches, our class allows us to choose positions with deviations tolerance levels deemed acceptable by regulators or investors. Another advantage is our measure represents how much an agent should shrink the position for it to be acceptable. Furthermore, considering a set with suitable properties, one can induce any positive homogeneous deviation measure. As a complementary result, we verify that the sub-level set of any deviation measure generates an acceptance set for deviation measures. For these measures there is no direct adaptation from the pre-existing acceptance sets of risk measures. These acceptance sets consider as acceptable positions with non positive risk, i.e., $\mathscr{A}_{\rho} = \{X : \rho(X) \leq 0\}$ (see Artzner et al. (1999), Delbaen (2002), Frittelli and Scandolo (2006), and Artzner et al. (2009)). This lack is mainly related to the axiomatic set of deviation measures. The classical approach requires monetary measures, i.e., a risk measure that fulfills translation invariance and monotonicity. However, deviation measures fulfill translation insensitivity and non-negativity. We have that the direct replacement of ρ by $\mathscr{D}$ in $\mathscr{A}_{\rho}$, i.e., $\mathscr{A}_{\rho} = \{X : \mathscr{D}(X) \leq 0\}$, is too restrictive due to non-negativity. As it only considers constant positions as acceptable. In addition, by virtue translation insensitivity the induced deviation is ∞ for any $X \notin \mathscr{A}_{\rho}$ and equal 0 if $X \in \mathscr{A}_{\rho}$.

The remainder of this work is structured in this format: Chapter 2 presents the notation, definitions,and preliminaries from the literature. Chapter 3 defines the set-induced deviations and exposes results regarding its continuity and financial theoretical properties. Furthermore, it presents how operations on the set affect the set-induced deviation. Chapter 4 defines the acceptance set induced by deviation measures, shows the relationship of properties of deviation measures with it and how operations with deviation measures reflect in its acceptance set. Also, it presents examples of acceptance sets induced by deviation measures for traditional deviation measures of financial literature. Finally, chapter 5 presents the final considerations.

2 Preliminaries

The content of this paper is based on the following notation. Consider the random result X of any asset, where $X \geq 0$ is a gain and $X \leq 0$ is a loss, that is defined in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We assume that financial positions represented by X are discounted by a risk-free rate. This assumption is standard in the risk management literature. All equalities and inequalities are considered almost surely in $\mathbb{P}$. We work on the space of random variables $L^p := L^p(\Omega, \mathscr{F}, \mathbb{P}), p \in [1, \infty]$, defined by the norm $\|X\|_p = (\mathbb{E}[|X|^p])^{\frac{1}{p}}$ when $p < \infty$, and $\|X\|_\infty = \inf\{w \in \mathbb{R} : |X| \leq w\}$ when $p = \infty$. Thereby, $X \in L^p$ represents that $\|X\|_p < \infty$. $\mathbb{E}[X]$ is the expected value of X under $\mathbb{P}$. $\{X_n\}, n \in \mathbb{N}$, represents a sequence. $X_n \to X$ denotes the convergence in the norm.

We define that a sequence of sets $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ decreases to $\mathscr{A}, \mathscr{A}_n \downarrow \mathscr{A}$, if $\mathscr{A} \subseteq \mathscr{A}_{n+1} \subseteq \mathscr{A}_n$ and $\bigcap_n \mathscr{A}_n = \mathscr{A}$. Moreover, $\{\mathscr{A}_n\}_{n\in\mathbb{N}}$ increases to $\mathscr{A}$, $\mathscr{A}_n \uparrow \mathscr{A}$, if $\mathscr{A}_n \subseteq \mathscr{A}_{n+1} \subseteq \mathscr{A}$ and $\bigcup_n \mathscr{A}_n = \mathscr{A}$. A sub-level set of a functional f is denoted by $\mathscr{A}_f^k \equiv \{X \in L^p : f(X) \leq k\}$. By abuse of notation, we identify a real number with the random variable that is almost-sure identical to it. We assume that for all families of sets $\{\mathscr{A}^k\}_{k\in\mathbb{R}}$, $0 \in \bigcap\{\mathscr{A}^k\}_{k\in\mathbb{R}}$. We define $\mathscr{A} + \mathscr{A}' = \{X + Y : X \in \mathscr{A}, Y \in \mathscr{A}'\}$ and for a real number $\lambda, \lambda\mathscr{A} = \{\lambda X : X \in \mathscr{A}\}$. We consider a deviation measure as a mapping $\mathscr{D} : L^p \to \mathbb{R}_+ \cup \{\infty\}$.

We begin by defining the properties of sets. There are a large number of possible properties. We present those that are the most relevant to this study.

Definition 2.1. *A set $\mathscr{A} \subset L^p$ may fulfills the following properties:*

(i) Law invariance: $\mathscr{A}$ is law invariant if $X \in \mathscr{A}$ and $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x), \forall\, x \in \mathbb{R}$, implies that $Y \in \mathscr{A}$, $\forall\, X, Y \in L^p$.

(ii) Closedness for scalar addition: $\mathscr{A}$ is closed for scalar addition if $\forall\, X \in \mathscr{A}$, we have that $X + c \in \mathscr{A}, \forall\, c \in \mathbb{R}$.

(iii) Radial boundedness: $\mathscr{A}$ is radially bounded if $\forall\, X \in \mathscr{A} \setminus \{0\}$ there is some $\delta_X \in (0, \infty)$, such that $\delta X \notin \mathscr{A}$, $\forall\, \delta \in (\delta_X, \infty)$. $\mathscr{A}$ is radially bounded at non constants if $\mathscr{A} \setminus \mathbb{R}$ is radially bounded.

(iv) Radiality: $\mathscr{A}$ is radial at some point $k \in \mathscr{A}$ if, for every $X \in \mathscr{A}$, there is some $\delta_X > 0$, such that for every $t \in [0, \delta_X]$ results in $k + tX \in \mathscr{A}$. When a set is radial at 0, it is absorbing.

(v) Convexity: $\mathscr{A}$ is convex if $\forall\, X, Y \in \mathscr{A}$ we have that $\lambda X + (1 - \lambda Y) \in \mathscr{A}$, $\forall\, \lambda \in [0, 1]$.

(vi) Star shapedness: $\mathscr{A}$ is star shaped if $\lambda X \in \mathscr{A}, \forall\, X \in \mathscr{A}$, $\forall\, \lambda \in [0, 1]$.

(vii) Positive homogeneity family: A family $\left\{\mathscr{A}^k\right\}_{k\in\mathbb{R}}$ is positive homogeneous if $\lambda\mathscr{A}^k = \mathscr{A}^{\lambda k}, \forall\, \lambda > 0$.

(viii) *Monotonic family: A family* $\{\mathscr{A}^k\}_{k\in\mathbb{R}}$ *is monotone if* $k \leq k'$ *implies in* $\mathscr{A}^k \subseteq \mathscr{A}^{k'}, \forall\, k, k' \in \mathbb{R}$.

Law Invariance indicates that if two random variables have the same distribution, then one is in the set if, and only if, the other is. Closedness for scalar addition indicates that the set is insensible to the addition of constants. This property indicates whether a set is not empty, then it contains all constants. Radially boundedness implies that whenever a ray passes through some points in the set, it eventually leaves it. In our framework, it means it is impossible to infinitely expand the position and keep its deviation acceptable. However it is possible to arbitrarily expand a constant position and it will not leave the real line. However, as the real line is not radially bounded, radially boundedness and closedness for scalar addition are incompatible. Thus, we introduce radially boundedness at non constants. Radiality at k means that part of the line segment emanating from k, to any random variable, lies in the set. When the set is absorbing and X is a financial position, it becomes possible to shrink any position until it "fits" the set. If a set is radially bounded at non constants and radial at its constants, then any ray that goes through a constant and any element of $L^p/\mathbb{R}$ has a segment in the set and another in its complement. Convexity indicates that any line connecting two points in the set still is in the set. Star shapedness is a weaker form of convexity. From this property if any X is in the set, then a scaled down X is as well. A desirable property, as it can intuitively mean that if an agent accepts to invest a certain amount in a stock, it also finds it acceptable to invest less in the same stock. Positive homogeneity family indicates that multiplication by a positive scalar is translated to the family index i.e. $k\mathscr{A}^1 = \mathscr{A}^k$. While monotonic family indicates that the family $\{\mathscr{A}^k\}_{k\in\mathbb{R}}$ is crescent in k.

Remark 2.2. Observe that the properties of radiality, convexity and closedness for scalar addition are kept under closure. Besides that, if the set is solid, i.e., $X \in \mathscr{A}, |Y| \leq |X|$ implies in $Y \in \mathscr{A}, \forall\, X, Y \in L^p$, and convex, then radial boudedness is also kept under closure. For a detailed proof see Lemma 3.2 of Koch-Medina et al. (2018). Note as well that if a set contains the origin and is convex, then it is star shaped.

We now define properties that a deviation measures may fulfill.

Definition 2.3. *A deviation measure* $\mathscr{D} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ *may fulfill the following properties, valid* $\forall\, X, Y \in L^p$*:*

(i) *Translation insensitivity:* $\mathscr{D}(X + c) = \mathscr{D}(X),\ \forall\, c \in \mathbb{R}$.

(ii) *Non-negativity:* $\mathscr{D}(X) > 0$ *for any non constant* X *and* $\mathscr{D}(X) = 0$ *for any constant* X.

(iii) *Positive homogeneity:* $\mathscr{D}(\lambda X) = \lambda \mathscr{D}(X), \forall\, \lambda \geq 0$.

(iv) *Convexity:* $\mathscr{D}(\lambda X + (1-\lambda)Y) \leq \lambda \mathscr{D}(X) + (1-\lambda)\mathscr{D}(Y), \forall\, \lambda \in [0,1]$.

(v) *Law invariance: If* $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x), \forall x \in \mathbb{R}$, *then* $\mathscr{D}(X) = \mathscr{D}(Y)$.

(vi) *Lower semi-continuity: If* $\{X_n\} \in L^p$, $X_n \to X$, *then* $D(X) \leq \liminf \mathscr{D}(X_n)$.

(vii) *Upper semi-continuity: If* $\{X_n\} \in L^p$, $X_n \to X$, *then* $D(X) \geq \limsup \mathscr{D}(X_n)$.

We refer as proper deviation measure a functional that respects translation insensitivity and non-negativity. When $\mathscr{D}$ *is a proper deviation measure and it respects convexity, it is a convex deviation in the sense proposed by Pflug (2006). When a convex deviation measure satisfies positive homogeneity, it is a generalized deviation measure in the sense of Rockafellar et al. (2006a). A deviation measure is said to be positive homogeneous, law invariant, lower semi-continuous or upper semi-continuous if it fulfills the properties of positive homogeneity, law invariance, lower semi-continuity or upper semi-continuous, respectively.*

Translation insensitivity, ensures that the deviation does not change if a constant value is added. Non-negativity implies that no constant assets have positive deviation and constant assets have deviation equal to 0. Positive homogeneity shows the deviation proportionally increases with the position size. Convexity ensures that diversification reduces the risk. Law invariance, indicates that two positions with the same distribution function have the same deviation.

3 Set-induced deviations

In this section, we define the set-induced deviation. We relate some of their continuity properties with the characteristics of the set. Continuity properties are important because deviation measures are functionals that require these properties to guarantee certain mathematical results, such as optimal values. We present the theoretical properties of set-induced deviations related to their generating set. These properties are important in identifying the validity and practical utility of our class in financial problems. We demonstrate how operations on the set affect them.

Definition 3.1. *Let $\mathscr{A} \subset L^p$. The set-induced deviations are functionals $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}^+$ defined as:*

$$\mathscr{D}_{\mathscr{A}}(X) = \inf\left\{m \in \mathbb{R}_+ \cup \{\infty\} : \frac{X}{m} \in \mathscr{A}\right\}. \tag{3.1}$$

$\mathscr{D}_{\mathscr{A}}$ can be understood as the amount that an agent should shrink a position for it to be acceptable, i.e., it is in the set.

Lemma 3.2. *Let $\mathscr{A}, \mathscr{A}' \subset L^p$ be star shaped sets, and $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$, $\mathscr{D}_{\mathscr{A}'} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be the set-induced deviation defined as in definition 3.1. If $\mathscr{A} \subseteq \mathscr{A}'$, then $\mathscr{D}_{\mathscr{A}}(X) \geq \mathscr{D}_{\mathscr{A}'}(X), \forall\, X \in L^p$. Furthermore, $X \in \mathscr{A}$ if, and only if, $\mathscr{D}_{\mathscr{A}}(X) \in [0,1]$.*

Proof. Consider that for each $X \in L^p$, we have a $M^X_{\mathscr{A}} \in \mathbb{R}^*_+$ such that $\frac{X}{m} \in \mathscr{A}$, $\forall\, m \geq M^X_{\mathscr{A}}$, and a $M^X_{\mathscr{A}'} \in \mathbb{R}^*_+$ such that $\frac{X}{m} \in \mathscr{A}'$, $\forall\, m \geq M^X_{\mathscr{A}'}$. Because $\mathscr{A} \subseteq \mathscr{A}'$, we get that every $\frac{X}{m} \in \mathscr{A}$ is also in $\mathscr{A}'$. Hence, $M^X_{\mathscr{A}} \geq M^X_{\mathscr{A}'}$. Thus,

$$\mathscr{D}_{\mathscr{A}}(X) = \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\} = \inf[M^X_{\mathscr{A}}, \infty) = M^X_{\mathscr{A}}$$

holds. Moreover, if there is no such $M^X_{\mathscr{A}}$ and $M^X_{\mathscr{A}'} \in \mathbb{R}^*_+$, then $D_A(X) = \mathscr{D}_{\mathscr{A}'}(X) = \infty$. Clearly, if there is a $M^X_{\mathscr{A}} \in \mathbb{R}$, so there is a $M^X_{\mathscr{A}'} \in \mathbb{R}$ and if only $M^X_{\mathscr{A}'} \in \mathbb{R}$ exists, we have that $\infty = \mathscr{D}_{\mathscr{A}}(X) > \mathscr{D}_{\mathscr{A}'}(X)$.

Note that if $X \in \mathscr{A}$ and $\mathscr{A}$ is star shaped, $\frac{X}{m} \in \mathscr{A}$, $\forall\, m \geq 1$, then $M^X_{\mathscr{A}} \leq 1$. Now, perceive that as the set is star shaped, $X \notin \mathscr{A}$ implies that $\lambda X \notin \mathscr{A}$, $\forall \lambda \geq 1$. We can rewrite λX as X'. Thus, if $X' \in \mathscr{A}$ we have $\frac{1}{\lambda}X'$, which by assumption is can not be in $\mathscr{A}$. Moreover, if $X \notin \mathscr{A}$, then $\frac{X}{m} \notin \mathscr{A}$, $\forall\, m \leq 1$. Hence, $M^X_{\mathscr{A}} > 1$. □

Proposition 3.3. *Let $\mathscr{A} \subset L^p$ be star shaped and absorbing set, and $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be a set-induced deviation defined as in definition 3.1. Then, we have:*

(i) If $\mathscr{A}$ is closed, then $\mathscr{D}_{\mathscr{A}}$ is lower semi-continuous.

(ii) If $\mathscr{A}$ is open, then $\mathscr{D}_{\mathscr{A}}$ is upper semi-continuous.

(iii) If $\left\{\mathscr{A}^k\right\}_{k\in\mathbb{N}}$ is a monotone sequence converging to $\mathscr{A}$, then $\mathscr{D}_{A^k}(X) \to \mathscr{D}_A(X), \forall\, X \in L^p$.

Proof. Concerning (i), if $\mathscr{A}$ is closed, $X_n \to X, \forall\, n \in \mathbb{N}$, then there is a M_{X_n} such that $\frac{X_n}{m} \in \mathscr{A}$, $\forall\, m \in [M_{X_n}, \infty), \forall\, n \in \mathbb{N}$, and a M_X such that $\frac{X}{m} \in \mathscr{A}$, $\forall\, m \in [M_X, \infty)$. Because the set is closed, we have that $\frac{X_n}{m} \in \mathscr{A}$ implies in $\frac{X}{m} \in \mathscr{A}$. Therefore, $[M_{X_n}, \infty) \subseteq [M_X, \infty), \forall\, n \in \mathbb{N}$ and $\lim_{(n\to\infty)}[M_{X_n}, \infty) \subseteq [M_X, \infty)$. Consequently,

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}^k\right\} \\
&= \inf[M_X, \infty) \leq \inf\left\{\lim_{(n\to\infty)} [M_{X_n}, \infty)\right\} \\
&= \inf\left\{\lim_{(n\to\infty)} \left\{m \in \mathbb{R}_+ : \frac{X_n}{m} \in \mathscr{A}\right\}\right\} \\
&\leq \liminf \mathscr{D}_{\mathscr{A}}(X_n).
\end{aligned}$$

For (ii), if $\mathscr{A}$ is open, $X_n \to X, \forall\, n \in \mathbb{N}$, then for all limit point $X \in \mathscr{A}$ there is an open ball in $\mathscr{A}$ that contains X_n, $\forall\, n \geq N$. By the same logic as before, when $\frac{X}{m} \in \mathscr{A}$ and $\frac{X_n}{m} \in \mathscr{A}$, $\forall\, n \geq N$, we get $[M_X, \infty) \subseteq \lim_{(n\to\infty)}[M_{X_n}, \infty)$ and $\limsup \mathscr{D}_{\mathscr{A}}(X_n) \leq \mathscr{D}_{\mathscr{A}}(X)$.

Relative to (iii), we have, by Lemma 3.2, that $\mathscr{A}^k \subseteq \mathscr{A}^{k'}$ implies in $\mathscr{D}_{\mathscr{A}^k} \geq \mathscr{D}_{\mathscr{A}^{k'}}$. Thereby, $\mathscr{A}^k \downarrow \mathscr{A}$ results that $\mathscr{D}_{\mathscr{A}^k}$ increases up to $\mathscr{D}_{\mathscr{A}}$, and $\mathscr{A}^k \uparrow \mathscr{A}$ implies that $\mathscr{D}_{\mathscr{A}^k}$ decreases to $\mathscr{D}_{\mathscr{A}}$. Either way, we obtain $\lim_{k\to\infty} \mathscr{D}_{\mathscr{A}^k}(X) = \mathscr{D}_{\mathscr{A}}(X), \forall\, X \in L^p$. □

Proposition 3.4. *Let $\mathscr{A} \subset L^p$ be star shaped , and $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be a set-induced functional defined as in definition 3.1. Then, we have the following:*

(i) $\mathscr{A} = \mathscr{A}^1_{\mathscr{D}_{\mathscr{A}}}$ and $\frac{X}{\mathscr{D}_{\mathscr{A}}(X)} \in bd(\mathscr{A}), \forall\, 0 < \mathscr{D}_{\mathscr{A}}(X) < \infty, \forall X \in L^p$, i.e. if $\mathscr{D}_{\mathscr{A}}(X) \in \mathbb{R}^*_+$, $\frac{X}{\mathscr{D}_{\mathscr{A}}(X)}$ is in the boundary of $\mathscr{A}$.

(ii) if $\mathscr{A}$ is absorbing, then $\mathscr{D}_{\mathscr{A}}$ is finite,

(iii) If $\mathscr{A}$ is closed and absorbing, then $\inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\} = \min\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}$, i.e., the minimum is attained, $\forall\, X \in L^p$.

(iv) If $\mathscr{A}$ is radially bounded, then $\mathscr{D}_{\mathscr{A}}(X) > 0, \forall\, X \in L^p$.

Proof. For (i), by Lemma 3.2, $\mathscr{A} = \mathscr{A}^1_{f_{\mathscr{A}}}$ is clear. To verify that $\frac{X}{\mathscr{D}_{\mathscr{A}}(X)} \in \mathrm{bd}(\mathscr{A})$ we must find that every ball around $\frac{X}{\mathscr{D}_{\mathscr{A}}(X)}$ has at least one point in $\mathscr{A}$ and one in $L^p \backslash \mathscr{A}$. For such, note that due to star shapedness, if $m \geq \mathscr{D}_{\mathscr{A}}(X), \frac{X}{m} \in \mathscr{A}$ and due to its construction $\mathscr{D}_{\mathscr{A}}(X) \geq m, \forall\, m$: $\frac{X}{m} \in \mathscr{A}, m > 0$. Therefore, if $0 \leq m < \mathscr{D}_{\mathscr{A}}(X), \frac{X}{m} \notin \mathscr{A}$. Take a sequence $\{M_n\} \in \mathbb{R}_+$, such that $M_n \uparrow \mathscr{D}_{\mathscr{A}}(X)$, then, $\forall\, m \in \{M_n\}, \frac{X}{m} \notin \mathscr{A}$. And for a sequence such that $N_n \downarrow \mathscr{D}_{\mathscr{A}}(X), \frac{X}{m} \in \mathscr{A}, \forall m \in \{N_n\}$. Thus $\frac{X}{\mathscr{D}_{\mathscr{A}}(X)} \in \mathrm{bd}(\mathscr{A})$.

Regarding (*ii*), we have that, by the absorbing property, there is always some $\delta_X \in \mathscr{A},\ \forall\, \delta \in [0, \delta_X), \forall\, X \in L^p$. It is straightforward to see that $\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}$ is never empty. Therefore, $\mathscr{D}_{\mathscr{A}}$ is finite.

Relative to (*iii*) Observe that for each $X \in L^p, \left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\} = [m_X, \infty)$. Thus, we have that the set is closed and bounded from below. Hence, we verify that the infimum is attained.

For (*iv*),if $\mathscr{A}$ is radially bounded then for each X there is a $M_X > 0$ such that $\frac{1}{m}X \notin \mathscr{A}, \forall m < M_X$. Therefore, $\inf\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\} > 0$. □

For sets that do not fulfill the star shapedness and closedness for scalar addition is possible to coerce it to respect these properties.

Remark 3.5. Let $\mathscr{B}$ be any set $\mathscr{A}_{\mathscr{B}} = \{Y : Y = \lambda X + c, X \in \mathscr{B}, c \in \mathbb{R}, \lambda \in [0,1]\}$. In this approach, if we consider the set $\mathscr{B}$ as the set containing some desirable positions, then the set $\mathscr{A}$ contains the shifted and downscaled versions of each position in $\mathscr{B}$. Thus, we have that $\mathscr{D}_{\mathscr{A}_{\mathscr{B}}}(X)$ assume values in $[0,1]$ for $X \in \mathscr{A}_{\mathscr{B}}$ and ∞ if $\lambda X \notin \mathscr{A}_{\mathscr{B}}, \forall \in \lambda \in \mathbb{R}$.

Lemma 3.6. *For all $\mathscr{A} \subset L^p$ and all $m, n \in \mathbb{R}_+$, $(m+n)\mathscr{A} \subseteq m\mathscr{A} + n\mathscr{A}$. If $\mathscr{A}$ is a convex set, $(m+n)\mathscr{A} = m\mathscr{A} + n\mathscr{A}$*

Proof. First, note that $(m+n)\mathscr{A} = \{(m+n)X : X \in \mathscr{A}\}$ and

$$\begin{aligned} m\mathscr{A} + n\mathscr{A} &= \{mX + nY : X, Y \in \mathscr{A}\} \\ &= \{mX + nY : X, Y \in \mathscr{A}, X = Y\} \cup \{mX + nY : X, Y \in \mathscr{A}, X \neq Y\} \\ &= \{mX + nX : X \in \mathscr{A}\} \cup \{mX + nY : X, Y \in \mathscr{A}, X \neq Y\} \\ &= \{(m+n)X : X \in \mathscr{A}\} \cup \{mX + nY : X, Y \in \mathscr{A}, X \neq Y\} \\ &= (m+n)\mathscr{A} \cup \{mX + nY : X, Y \in \mathscr{A}, X \neq Y\} \\ &\supseteq (m+n)\mathscr{A}. \end{aligned}$$

Now, assume $\mathscr{A}$ is convex then, $\forall\, X \in m\mathscr{A} + n\mathscr{A}, X = mY + nY', Y, Y' \in \mathscr{A}$. Hence

$$X = mY + nY' = (m+n)\left(\frac{m}{m+n}Y + \frac{n}{m+n}Y'\right)$$

as $\frac{m}{m+n} + \frac{n}{m+n} = 1$, it is a convex combination. Then $\frac{m}{m+n}Y + \frac{n}{m+n}Y' \in \mathscr{A}$ and $(m+n)\left(\frac{m}{m+n}Y + \frac{n}{m+n}Y'\right) \in (m+n)\mathscr{A}$. □

Proposition 3.7. *Let $\mathscr{A} \subset L^p$ and $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be the set-induced deviation defined as in 3.1. Then, we have:*

(i) $\mathscr{D}_{\mathscr{A}}$ respects positive homogeneity.

(ii) *If* $\mathbb{R} \subset \mathscr{A}$ *and* $\mathscr{A}$ *is radially bounded at non constants, then* $\mathscr{D}_{\mathscr{A}}$ *is non negative.*

(iii) *If* $\mathscr{A}$ *is closed for scalar addition, then* $\mathscr{D}_{\mathscr{A}}$ *is translation insensitive.*

(iv) *If* $\mathscr{A}$ *is convex, then* $\mathscr{D}_{\mathscr{A}}$ *is convex.*

(v) *If* $\left\{\mathscr{A}^k\right\}_{k\in\mathbb{R}_+^*}$ *is a family of positive homogeneous sets, then* $k\mathscr{D}_{\mathscr{A}^k} = k'\mathscr{D}_{\mathscr{A}^{k'}}, \forall\, k,k' \in \mathbb{R}_+^*$ *and* $k\mathscr{D}_{\mathscr{A}^k}(X) = \inf\{m \in \mathbb{R}_+ : X \in \mathscr{A}^m\}$.

(vi) *If* $\mathscr{A}$ *is law invariant, then* $\mathscr{D}_{\mathscr{A}}$ *is as well.*

Proof. For (*i*), $\forall\, \lambda \geq 0$, we have to

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}}(\lambda X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{\lambda X}{m} \in \mathscr{A}\right\} \\
&= \inf\left\{\lambda n \in \mathbb{R}_+ : \frac{X}{n} \in \mathscr{A}\right\} \\
&= \lambda \inf\left\{n \in \mathbb{R}_+ : \frac{X}{n} \in \mathscr{A}\right\} \\
&= \lambda \mathscr{D}_{\mathscr{A}}(X).
\end{aligned}$$

Thus, $\mathscr{D}_{\mathscr{A}}$ fulfills positive homogeneity.

Regarding (*ii*), let $\mathbb{R} \subset \mathscr{A}$, we have that $\left\{m \in \mathbb{R}_+ : \frac{c}{m} \in \mathscr{A}\right\} = \mathbb{R}_+, \forall\, c \in \mathbb{R}$. Therefore, for any $c \in \mathbb{R}$, we obtain $\inf\left\{m \in \mathbb{R}_+ : \frac{c}{m} \in \mathscr{A}\right\} = 0$. As we have that set is radially bounded at non constants, for any non constant $X \in \mathscr{A}$ there is some $m > 0$ such that $1/m < \infty$ and $\frac{1}{m}X \notin \mathscr{A}$. From this, it follows that $\inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\} > 0$. Hence, $\mathscr{D}_{\mathscr{A}}$ respects non-negativity.

In (*iii*), let $c \in \mathbb{R}$ and $X \in L^p$. Since $\mathscr{A}$ is closed for scalar addition, we have that

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}}(X+c) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X+c}{m} \in \mathscr{A}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} + \frac{c}{m} \in \mathscr{A}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\} \\
&= \mathscr{D}_{\mathscr{A}}(X).
\end{aligned}$$

Thereby, $\mathscr{D}_{\mathscr{A}}$ is translation insensitive.

Regarding (*iv*), let $\mathscr{A}_{\mathscr{D}}$ be a convex set, therefore, $(m+n)\mathscr{A} = m\mathscr{A} + n\mathscr{A}$, then,

$$
\begin{aligned}
\mathscr{D}_{\mathscr{A}}(X)+\mathscr{D}_{\mathscr{A}}(Y) &= \inf\left\{m\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}\right\}+\inf\left\{m\in\mathbb{R}_+ : \frac{Y}{m}\in\mathscr{A}\right\}\\
&= \inf\left\{m_X+m_Y\in\mathbb{R}_+ : \frac{X}{m_X}\in\mathscr{A},\frac{Y}{m_X}\in\mathscr{A}\right\}\\
&= \inf\{m_X+m_Y\in\mathbb{R}_+ : X\in\mathscr{A}m_X, Y\in\mathscr{A}m_Y\}\\
&\geq \inf\{m_X+m_Y\in\mathbb{R}_+ : X+Y\in\mathscr{A}m_X+\mathscr{A}m_Y\}\\
&= \inf\{m_X+m_Y\in\mathbb{R}_+ : X+Y\in\mathscr{A}(m_X+m_Y)\}\\
&= \inf\{m\in\mathbb{R}_+ : X+Y\in\mathscr{A}m\}\\
&= \inf\left\{m\in\mathbb{R}_+ : \frac{X+Y}{m}\in\mathscr{A}\right\}\\
&= \mathscr{D}_{\mathscr{A}}(X+Y).
\end{aligned}
$$

Hence, as $\mathscr{D}_{\mathscr{A}}$ is positive homogeneous, it is convex.

For (v), $\forall\, k,k'\in\mathbb{R}_+^*$, we have, due to it being a positive homogeneous family, that $\frac{k'}{k}\mathscr{A}^k=\mathscr{A}^{k'}$. Therefore,

$$
\begin{aligned}
k\mathscr{D}_{\mathscr{A}^k}(X) &= k\inf\left\{m\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}^k\right\}\\
&= k'\inf\left\{\frac{k}{k'}m\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}^k\right\}\\
&= k'\inf\left\{m\in\mathbb{R}_+ : \frac{k}{k'}\frac{X}{m}\in\mathscr{A}^k\right\}\\
&= k'\inf\left\{m\in\mathbb{R}_+ : \frac{X}{m}\in\frac{k'}{k}\mathscr{A}^k\right\}\\
&= k'\inf\left\{m\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}^{k'}\right\}\\
&= k'\mathscr{D}_{\mathscr{A}^{k'}}(X).
\end{aligned}
$$

Because k and k' are arbitrarily chosen the assertion holds. To rewrite the set-induced deviation $k\mathscr{D}_{\mathscr{A}^k}$ as $\mathscr{D}_{\mathscr{A}}(X)=\inf\{m\in\mathbb{R}_+ : X\in\mathscr{A}^m\}$ we have:

$$
\begin{aligned}
k\mathscr{D}_{\mathscr{A}^k} &= \inf\left\{km\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}^k\right\}\\
&= \inf\left\{m\in\mathbb{R}_+ : \frac{kX}{m}\in\mathscr{A}^k\right\}\\
&= \inf\left\{m\in\mathbb{R}_+ : X\in\mathscr{A}^{\frac{mk}{k}}\right\}\\
&= \inf\{m\in\mathbb{R}_+ : X\in\mathscr{A}^m\}.
\end{aligned}
$$

For (vi), let $X\in L^p$ and $Y\in L^p$ have the same distribution and $m\in\mathbb{R}_+$. From this, we have to $\frac{X}{m}$ also follows the same distribution of $\frac{Y}{m}$. This leads to

$$
\mathscr{D}_{\mathscr{A}}(X)=\inf\left\{m\in\mathbb{R}_+ : \frac{X}{m}\in\mathscr{A}\right\}=\inf\left\{m\in\mathbb{R}_+ : \frac{Y}{m}\in\mathscr{A}\right\}=\mathscr{D}_{\mathscr{A}}(Y).
$$

□

Remark 3.8. (Theorem 1 of Rockafellar et al. (2006a)). If $\mathscr{A}$ is a radially bounded at non constants, closed for scalar addition, convex and closed, then $\mathscr{D}_{\mathscr{A}}$ is a lower-semi-continuous generalized deviation measure and admits the following representation:

$$\mathscr{D}_{\mathscr{A}}(X) = \mathbb{E}[X] - \inf_{\mathbb{Q} \in \mathscr{Q}} \mathbb{E}[X\mathbb{Q}],$$

where $\mathscr{Q} \subset L^q$ is the risk envelop uniquely determined by $\mathscr{D}_{\mathscr{A}}$:

$$\mathscr{Q} = \{\mathbb{Q} \in L^q : \mathscr{D}_{\mathscr{A}}(X) \geq \mathbb{E}[X] - \mathbb{E}[X\mathbb{Q}], \forall\, X \in L^p\}.$$

Where when $p < \infty$, $q = \frac{p}{p-1}$, and L^q is the dual space of L^p and when $p = \infty, q = 1$. $\mathscr{Q}$ is non empty, closed and convex. For every non constant $X \in L^p$, $\exists\, \mathbb{Q} \in \mathscr{Q}$, such that $\mathbb{E}[X\mathbb{Q}] < \mathbb{E}[X]$, and $\mathbb{E}[\mathbb{Q}] = 1, \forall\, \mathbb{Q} \in \mathscr{Q}$.

We now show how some operations on the sets affect the set-induced deviation.

Proposition 3.9. *Let $\mathscr{A} \subset L^p$ and $\mathscr{B} \subset L^p$ be star shaped and absorbing sets, $\mathscr{D}_{\mathscr{A}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ and $\mathscr{D}_{\mathscr{B}} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be set-induced deviations defined as in 3.1 and $a, b \in L^p$ Then, we have the following $\forall\, X, Y \in L^p$:*

(i) $\mathscr{D}_{\mathscr{A}\cup\mathscr{B}}(X) = \min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X))$.

(ii) $\mathscr{D}_{\mathscr{A}\cap\mathscr{B}}(X) = \max(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X))$.

(iii) $D_{\mathscr{A}\setminus\mathscr{B}}(X) = \begin{cases} \mathscr{D}_{\mathscr{A}}(X), \text{ if } \mathscr{D}_{\mathscr{A}}(X) < \mathscr{D}_{\mathscr{B}}(X). \\ \infty, \text{ otherwise.} \end{cases}$

(iv) $D_{\mathscr{A}\Delta\mathscr{B}}(X) = \begin{cases} \infty, \text{ if } \mathscr{D}_{\mathscr{A}}(X) = \mathscr{D}_{\mathscr{B}}(X). \\ \min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)), \text{ otherwise.} \end{cases}$

(v) $\mathscr{D}_{\mathscr{A}+\mathscr{B}}(X) = \inf\{m \in \mathbb{R}_+ : X = m(a+b), \mathscr{D}_{\mathscr{A}}(a) \leq 1, \mathscr{D}_{\mathscr{B}}(b) \leq 1\}$.

(vi) $\mathscr{D}_{\lambda\mathscr{A}}(X) = \dfrac{\mathscr{D}_{\mathscr{A}}(X)}{\lambda}, \forall\, \lambda \in \mathbb{R}_+^*$.

Proof. By Proposition 3.3 we have that $\mathscr{A} = \mathscr{A}^1_{\mathscr{D}_{\mathscr{A}}}$ and by Lemma 3.2 we that that if $X \in \mathscr{A}$, then $\mathscr{D}_A(X) \leq 1$.

For (i), we get

$$\begin{aligned} \mathscr{D}_{\mathscr{A}\cup\mathscr{B}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A} \cup \mathscr{B}\right\} \\ &= \min\left(\inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}, \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{B}\right\}\right) \\ &= \min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)). \end{aligned}$$

For (*ii*), we have that

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}\cap\mathscr{B}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\cap\mathscr{B}\right\} \\
&= \max\left(\inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}, \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{B}\right\}\right) \\
&= \max\left(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)\right).
\end{aligned}$$

In (*iii*) as by Proposition 3.7, $D_{\mathscr{A}}$ and $\mathscr{D}_{\mathscr{B}}$ are positive homogeneous, we have to

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}\setminus\mathscr{B}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\setminus\mathscr{B}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : \frac{\mathscr{D}_{\mathscr{A}}(X)}{m} \leq 1 < \frac{\mathscr{D}_{\mathscr{B}}(X)}{m}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : \mathscr{D}_{\mathscr{A}}(X) \leq m < \mathscr{D}_{\mathscr{B}}(X)\right\} \\
&= \mathscr{D}_{\mathscr{A}}(X), \text{ if } \mathscr{D}_{\mathscr{A}}(X) < \mathscr{D}_{\mathscr{B}}(X).
\end{aligned}$$

Otherwise, $\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\setminus\mathscr{B}\right\}$ is empty and $\mathscr{D}_{\mathscr{A}\setminus\mathscr{B}}(X) = \infty$.

Concerning (*iv*), the symmetric difference can be written as $(\mathscr{A}\cup\mathscr{B})\setminus(\mathscr{A}\cap\mathscr{B})$, which by (*i*) and (*iii*) allow us to write as

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}\Delta\mathscr{B}}(X) &= \mathscr{D}_{(\mathscr{A}\cup\mathscr{B})\setminus(\mathscr{A}\cap\mathscr{B})}(X) \\
&= \begin{cases} \min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)), \text{ if } \min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X) < \max(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)). \\ \infty, \text{ otherwise.} \end{cases}
\end{aligned}$$

The $\min(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X)) \geq \max(\mathscr{D}_{\mathscr{A}}(X), \mathscr{D}_{\mathscr{B}}(X))$ only is fulfilled when $\mathscr{D}_{\mathscr{A}}(X) = \mathscr{D}_{\mathscr{B}}(X)$. Therefore, the assertion holds.

For (*v*), we have the following

$$\begin{aligned}
\mathscr{D}_{\mathscr{A}+\mathscr{B}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}+\mathscr{B}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} = a+b, a \in \mathscr{A}, b \in \mathscr{B}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : X = m(a+b), a \in \mathscr{A}, b \in \mathscr{B}\right\} \\
&= \inf\left\{m \in \mathbb{R}_+ : X = m(a+b), \mathscr{D}_{\mathscr{A}}(a) \leq 1, \mathscr{D}_{\mathscr{B}}(b) \leq 1\right\}.
\end{aligned}$$

In regard to (*vi*), we get $\forall\, \lambda \in \mathbb{R}_+^*$

$$\begin{aligned}\mathscr{D}_{\lambda\mathscr{A}}(X) &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \lambda\mathscr{A}\right\}\\ &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{\lambda m} \in \mathscr{A}\right\}\\ &= \inf\left\{\frac{m}{\lambda} \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}\\ &= \frac{\mathscr{D}_{\mathscr{A}}(X)}{\lambda}.\end{aligned}$$

□

Remark 3.10. With the product vector space in mind we can define the set-induced deviation from the product space $L^p \times L^q$, where $q \geq 1$, into the extended real line, $\mathscr{D}_{\mathscr{A}\times\mathscr{B}} : L^p \times L^q \to \mathbb{R}_+ \cup \{\infty\}$ as similarly as the set-induced deviations from L^p as possible, in the following form:

$$\mathscr{D}_{\mathscr{A}\times\mathscr{B}}(X,Y) = \inf\left\{m \in \mathbb{R}_+ : \frac{(X,Y)}{m} \in \mathscr{A}\times\mathscr{B}\right\}, \forall\,(X,Y) \in L^p \times L^q, \forall\,\mathscr{A}\times\mathscr{B} \subset L^p \times L^q.$$

Then, if we have that $\mathscr{A} \subset L^p$ and $\mathscr{B} \subset L^q$ are star shaped and absorbing sets, we get that,

$$\mathscr{D}_{\mathscr{A}\times\mathscr{B}}(X,Y) = \max(\mathscr{D}_A(X), \mathscr{D}_{\mathscr{B}}(Y)), \forall\, X \in L^p, \forall\, Y \in L^q.$$

To see it, note that it following chain holds:

$$\begin{aligned}\mathscr{D}_{\mathscr{A}\times\mathscr{B}}(X,Y) &= \inf\left\{m \in \mathbb{R}_+ : \frac{(X,Y)}{m} \in \mathscr{A}\times\mathscr{B}\right\}\\ &= \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}, \frac{Y}{m} \in \mathscr{B}\right\}\\ &= \max\left(\inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}\right\}, \inf\left\{m \in \mathbb{R}_+ : \frac{Y}{m} \in \mathscr{B}\right\}\right)\\ &= \max(\mathscr{D}_A(X), \mathscr{D}_{\mathscr{B}}(Y)).\end{aligned}$$

4 Acceptance set induced by deviation measures

The sub-level set of any deviation measure generate a valid acceptance set for deviations measures. When a deviation measure is positive homogeneous, the set-induced deviation is a scaled version of the original measure. The set-induced deviations when the chosen set is the acceptance set induced by deviation measures are a special case of the approach proposed in the previous section. In this section, we define the acceptance set induced by deviation measures. We relate relevant properties of deviation measures with its acceptance set. We show how some operations of deviation measures reflect in the acceptance set induced by deviation measures. We also provide examples of acceptance sets induced by traditional deviation measures to illustrate the importance of the proposed approach.

Definition 4.1. *Let $\mathscr{D} : L^p \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be a deviation measure and $k \in \mathbb{R}_+^*$. The acceptance set induced by deviation measures at level k are defined as:*

$$\mathscr{A}_{\mathscr{D}}^k = \{X \in L^p : \mathscr{D}(X) \leq k\}. \tag{4.1}$$

$\mathscr{A}_{\mathscr{D}}^k$ defines the set of financial positions viewed as acceptable, given a coefficient of aversion to deviation k. Higher values of k indicate a higher risk tolerance, i.e., a greater tolerance to the deviation. For the agent to determine the minimum amount that must reduce the position, for the position to be contained in the set, one should employ the set-induced deviation. When set-induced deviations are induced by $\mathscr{A}_{\mathscr{D}}^k$ we refer to as $\mathscr{D}_{\mathscr{A}_{\mathscr{D}}^k}$.

Remark 4.2. We require that k greater than 0 because otherwise we have a set that contains only constants, possibly being empty. It is interesting to note that the acceptance sets are a monotone family. For a finite sequence N of real numbers, if $\max\{n \in N\} = n^*$ and $\min\{n \in N\} = n_*$, then $\bigcup_{n\in N} \mathscr{A}_{\mathscr{D}}^n = \mathscr{A}_{\mathscr{D}}^{n^*}$ and $\bigcap_{n\in N} \mathscr{A}_{\mathscr{D}}^n = \mathscr{A}_{\mathscr{D}}^{n_*}$. Therefore, for two different positions $X, Y \in L^p$ for which $\mathscr{D}(X) < \mathscr{D}(Y)$, we have m, n, where $n < m$, such that $Y \in \mathscr{A}_{\mathscr{D}}^m$ and $Y \notin \mathscr{A}_{\mathscr{D}}^n$. Thus, $X \in \mathscr{A}_{\mathscr{D}}^n \subset \mathscr{A}_{\mathscr{D}}^m$. Furthermore, as $\mathscr{A}_{\mathscr{D}}^k$ is a sub-level set, if the measure is lower semi-continuous, then the set is closed.

Proposition 4.3. *Let $\mathscr{D} : L^p \rightarrow \mathbb{R}_+ \cup \{\infty\}$ be a positive homogeneous deviation measure and $\mathscr{A}_{\mathscr{D}}^k$ be an acceptance set induced by deviation measure defined as in 4.1. Then, we have the following $\forall\, k, k' \in \mathbb{R}_+^*$:*

(i) *$\mathscr{A}_{\mathscr{D}}^k$ is star shaped, $\{\mathscr{A}_{\mathscr{D}}^k\}_{k\in\mathbb{R}_+}$ is a positive homogeneous family and $\mathscr{D}$ can be recovered from $\mathscr{A}_{\mathscr{D}}^k$ as follows:*

$$\mathscr{D}(X) = k \inf\left\{m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}_{\mathscr{D}}^k\right\} = \inf\left\{k \in \mathbb{R}_+ : X \in \mathscr{A}_{\mathscr{D}}^k\right\} = k\mathscr{D}_{\mathscr{A}_{\mathscr{D}}^k}(X), \forall\, X \in L^p.$$

(ii) If $\mathscr{D}$ is finite, then $\mathscr{A}^k_{\mathscr{D}}$ is absorbing.

(iii) If $\mathscr{D}$ is translation insensitive and finite, then $\mathscr{A}^k_{\mathscr{D}}$ is radial in $\mathbb{R}$ and closed for scalar addition.

(iv) If $\mathscr{D}$ is non-negative, then $\mathscr{A}^k_{\mathscr{D}}$ is radially bounded at non-constants and $\mathbb{R} \in \mathscr{A}^k$.

(v) If $\mathscr{D}$ is a convex functional, then $\mathscr{A}^k_{\mathscr{D}}$ is a convex set.

(vi) If $\mathscr{D}$ is law invariant, then $\mathscr{A}^k_{\mathscr{D}}$ is as well.

Proof. We begin in (i). The star shapedness is clear as if $\mathscr{D}(X) \leq k$, then $\lambda\mathscr{D}(X) \leq k$, $\forall\, \lambda \in [0,1]$. Also, note that

$$\lambda \mathscr{A}^k_{\mathscr{D}} = \{\lambda X \in L^p : \mathscr{D}(X) \leq k\} = \{X \in L^p : \mathscr{D}(X) \leq \lambda k\} = \mathscr{A}^{\lambda k}_{\mathscr{D}}$$

establishes the positive homogeneity for the generated family. Since we assume that $\mathscr{D}$ satisfies positive homogeneity and $k \in \mathbb{R}^*_+$, we get that

$$\begin{aligned} k\mathscr{D}_{\mathscr{A}^k_{\mathscr{D}}}(X) &= k \inf\left\{ m \in \mathbb{R}_+ : \frac{X}{m} \in \mathscr{A}^k_{\mathscr{D}} \right\} \\ &= k \inf\left\{ m \in \mathbb{R}_+ : \mathscr{D}\left(\frac{X}{m}\right) \leq k \right\} \\ &= k \inf\left\{ m \in \mathbb{R}_+ : \frac{\mathscr{D}(X)}{k} \leq m \right\} \\ &= \mathscr{D}(X). \end{aligned}$$

For the second equality, we have that

$$\inf\left\{ k \in \mathbb{R}_+ : X \in \mathscr{A}^k \right\} = \inf\{k \in \mathbb{R}_+ : \mathscr{D}(X) \leq k\} = \mathscr{D}(X).$$

Note that if $\mathscr{D}(X) = \infty$, then $\frac{X}{m} \notin \mathscr{A}^k$, $\forall\, k \in \mathbb{R}^*_+, \forall\, m \in \mathbb{R}_+$ and $\inf\{\emptyset\} = \infty$. This ensures that the previous equalities holds even in such case.

For (ii), due to finitedness and positive homogeneity, it is clear that for any X and k, there is a δ^k_X such that $\delta\mathscr{D}(X) \leq k$, $\forall\, \delta \in [0, \delta^k_X]$.

Regarding (iii), we have that if $\mathscr{D}$ respects translation insensitivity and positive homogeneity, then $\mathscr{D}(c) = 0, \forall\, c \in \mathbb{R}$, and $c \in \mathscr{A}^k_{\mathscr{D}}, \forall\, c \in \mathbb{R}$. To prove that $\mathscr{A}^k_{\mathscr{D}}$ is radial at any constant, it is enough to show there is some $\delta > 0$ such that $c + tX \in \mathscr{A}^k_{\mathscr{D}}$ whenever $0 < t \leq \delta$. Let $\delta = \frac{k}{\mathscr{D}(X)}$. Then, we have that

$$\mathscr{D}(c + tX) = t\mathscr{D}(X) \leq \frac{k}{\mathscr{D}(X)}\mathscr{D}(X) = k.$$

Referring to closedness for scalar addition, let $X \in \mathscr{A}^k_{\mathscr{D}}$, we have that $\mathscr{D}(X + c) = \mathscr{D}(X) \leq k$. This evidences that $X + c \in \mathscr{A}^k_{\mathscr{D}}, \forall\, c \in \mathbb{R}$. Besides that, let $X \notin \mathscr{A}^k_{\mathscr{D}}$, we have that $k < \mathscr{D}(X) = \mathscr{D}(X + c)$. This implies in $X + c \notin \mathscr{A}^k, \forall\, c \in \mathbb{R}$.

For (*iv*), it is straightforward to see that if $\mathscr{D}$ respects non-negativity, then for all non-constant X, there is some $\lambda_X > 0$ such that $\lambda\mathscr{D}(X) > k > 0$ for all $\lambda \in (\lambda_X, \infty)$. And as non-negativity implies that $\mathscr{D}(c) = 0, \forall\, c \in \mathbb{R}$ and $k > 0$ it is clear that $c \in \mathscr{A}^k_{\mathscr{D}}\ \forall\, c \in \mathbb{R}$.

Regarding (*v*), let $\mathscr{D}$ be convex, $X, Y \in \mathscr{A}^k_{\mathscr{D}}$ and $\lambda \in [0,1]$. Therefore, we get

$$\mathscr{D}(\lambda X + (1-\lambda)Y) \leq \lambda\mathscr{D}(X) + (1-\lambda)\mathscr{D}(Y) \leq k.$$

This implies in $\lambda X + (1-\lambda)Y \in \mathscr{A}^k_{\mathscr{D}}$.

For (*vi*), if $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x)$, then $\mathscr{D}(X) = \mathscr{D}(Y)$. We get $X \in \mathscr{A}^k_{\mathscr{D}}$ if, and only if, $Y \in \mathscr{A}^k_{\mathscr{D}}$. □

Remark 4.4. Note that when a acceptance set is closed to scalar addition, we have that $\inf\{m \in \mathbb{R} : m \in \mathscr{A}\} = -\infty$. Therefore, this property is incompatible with the traditional monetary acceptance set. Thus, a set that respects it can not induce a monetary risk measure. We suggest Artzner et al. (2009) and Chapter 4 of Föllmer and Schied (2016) for details about induced monetary risk measure.

Remark 4.5. When we want greater protection in our position, a viable way is to penalize by a monetary amount $c \in \mathbb{R}_+$ the acceptance set associated to some risk measure, i.e., $\mathscr{A}_\rho - c = \{X : \rho(X) \leq 0\} - c = \{X : \rho(X) \leq -c\}$, where ρ is a coherent risk measure[1]. We denote this set by $\mathscr{A}^c_\rho$. However, this set considers only the risk from the viewpoint of loss, that is, the possibility of a negative outcome. To take variability in account, we consider $\mathscr{A}^c_\rho \cap \mathscr{A}^k_{\mathscr{D}}$. One can extract a measure of this set as follows:

$$\inf\left\{m \in \mathbb{R} : \frac{X+m}{m_2} \in \mathscr{A}^c_\rho \cap \mathscr{A}^k_{\mathscr{D}}, m_2 \in \mathbb{R}^*_+\right\} = \rho(X) + \frac{c}{k}\mathscr{D}(X).$$

Using of Proposition 4.7 of Righi (2018), if $\frac{c}{k} \leq \inf\left\{\frac{-\inf X - \rho(X)}{\mathscr{D}(X)} : X \in L^p, \mathscr{D}(X) > 0\right\}$, then the composition is also coherent.

Now, we analyze how some operation on deviation measures reflect in its acceptance set.

Proposition 4.6. *Let $\mathscr{D} : L^p \to \mathbb{R}_+ \cup \{\infty\}$ and $\mathscr{D}' : L^p \to \mathbb{R}_+ \cup \{\infty\}$ be positive homogeneous deviation measures and $k, k' \in \mathbb{R}^*_+$. Then:*

(i) $\mathscr{A}^k_{\min(\mathscr{D},\mathscr{D}')} = \mathscr{A}^k_{\mathscr{D}} \cup \mathscr{A}^k_{\mathscr{D}'}$.

(ii) $\mathscr{A}^k_{\max(\mathscr{D},\mathscr{D}')} = \mathscr{A}^k_{\mathscr{D}} \cap \mathscr{A}^k_{\mathscr{D}'}$.

(iii) $\mathscr{A}^{k+k'}_{\mathscr{D}+\mathscr{D}'} \supseteq \mathscr{A}^k_{\mathscr{D}} \cap \mathscr{A}^{k'}_{\mathscr{D}'}$.

(iv) $\mathscr{A}^k_{\lambda\mathscr{D}} = \lambda\mathscr{A}^k_{\mathscr{D}}$.

[1] A coherent risk measure fulfills translation invariance, monotonicity, convexity and positive homogeneity. See Artzner et al. (1999).

Proof. For (*i*), if $X \in \mathscr{A}^k_{\min(\mathscr{D},\mathscr{D}')}$, then $\mathscr{D}(X) \leq k$ or $\mathscr{D}'(X) \leq k$. This implies that X is in either $\mathscr{A}^k_{\mathscr{D}}$ or in $\mathscr{A}^k_{\mathscr{D}'}$. Therefore, it is in the union. In addition, since $X \in \mathscr{A}^k_{\mathscr{D}} \cup \mathscr{A}^k_{\mathscr{D}'}$, so we have $\mathscr{D}(X) \leq k$ or $\mathscr{D}'(X) \leq k$. This indicates that $\min(\mathscr{D}(X), \mathscr{D}'(X)) \leq k$.

For (*ii*), if $X \in \mathscr{A}^k_{\max(\mathscr{D},\mathscr{D}')}$, then $\mathscr{D}(X) \leq k$ and $\mathscr{D}'(X) \leq k$. This implies that X is in $\mathscr{A}^k_{\mathscr{D}}$ and in $\mathscr{A}^k_{\mathscr{D}'}$. Therefore, it is in the intersection. If $X \in \mathscr{A}^k_{\mathscr{D}} \cap \mathscr{A}^k_{\mathscr{D}'}$, then $\mathscr{D}(X) \leq k$ and $\mathscr{D}'(X) \leq k$, which results that $\max(\mathscr{D}(X), \mathscr{D}'(X)) \leq k$.

Regarding (*iii*), if $X \in \mathscr{A}^k_{\mathscr{D}} \cap \mathscr{A}^{k'}_{\mathscr{D}'}$, then $\mathscr{D}(X) \leq k$ and $\mathscr{D}'(X) \leq k'$. Hence, $\mathscr{D}(X) + \mathscr{D}'(X) \leq k + k'$, which implies in $\mathscr{A}^{k+k'}_{\mathscr{D}+\mathscr{D}'}$.

Concerning (*iv*), if $X \in \mathscr{A}^k_{\lambda\mathscr{D}}$, then $\lambda\mathscr{D}(X) \leq k$ which, due to positive homogeneity, implies in $\lambda^{-1}X \in \mathscr{A}^k_{\mathscr{D}}$ and $X \in \lambda\mathscr{A}^k_{\mathscr{D}}$. Finally, $X \in \lambda\mathscr{A}^k_{\mathscr{D}}$ represents that $\mathscr{D}(X\lambda^{-1}) \leq k$. Therefore, $X \in \mathscr{A}^k_{\lambda\mathscr{D}}$. □

Example 4.7. We provide examples of possible choices of $\mathscr{A}^k_{\mathscr{D}}$, which are induced by well-known deviation measures. Let $X \in L^p$ and $k \in R^*_+$.

(i) Variance (σ^2): One of the most classical deviation measures. It is defined as

$$\sigma^2(X) = \|X - E[X]\|_2^2$$

Its acceptance set induced is defined as

$$\mathscr{A}^k_{\sigma^2} = \left\{X \in L^p : \sigma^2(X) \leq k\right\}.$$

As the only deviation measure that does not respect positive homogeneity, its set-induced deviation is not a scaled version of the variance itself. It is given by

$$\mathscr{D}_{\mathscr{A}^k_{\sigma^2}}(X) = \frac{\|X - E[X]\|_2}{\sqrt{k}}.$$

(ii) Standard deviation (σ): This measure is the inspiration for the whole class of generalized deviation measures. One can define it by

$$\sigma(X) = \|X - E[X]\|_2.$$

Its acceptance set induced is represented by

$$\mathscr{A}^k_{\sigma} = \left\{X \in L^p : \sigma^2(X) \leq (k)^2\right\}.$$

(iii) Standard lower semi-deviation (σ_-): It is a generalized deviation measure that considers only the underpart of the deviation. This measure is defined by

$$\sigma_-(X) = \|(X - E[X])_-\|_2.$$

Its acceptance set induced is defined as follows

$$\mathscr{A}^k_{\sigma_-} = \left\{ X \in L^p : \sigma^2(X|\mathbb{E}[X] \geq X) \leq \frac{k^2}{\mathbb{P}(\mathbb{E}[X] \geq X)} \right\},$$

where $\sigma^2(X|\mathbb{E}[X] \geq X) = \mathbb{E}[(X - \mathbb{E}[X|\mathbb{E}[X] \geq X])^2 \mid \mathbb{E}[X] \geq X]$ is the conditional variance. This set also contains every random variables whose variance is bounded by $(k)^2$, but as $\mathbb{P}(\mathbb{E}[X] \geq X) \in [0,1]$ the conditional variance is always smaller than the full variance. We have that $\mathscr{A}^k_{\sigma} \subseteq \mathscr{A}^k_{\sigma_-}$. Hence, this one is less restrictive than the one above.

(iv) Standard upper semi-deviation (σ_+): This measure is similar to' the standard lower semi-deviation by changing the negative for the positive part. It is defined as

$$\sigma_+(X) = \|(X - E[X])_+\|_2.$$

Its acceptance set induced is defined conform

$$\mathscr{A}^k_{\sigma_+} = \left\{ X \in L^p : \sigma^2(X|\mathbb{E}[X] \leq X) \leq \frac{k^2}{\mathbb{P}(\mathbb{E}[X] \leq X)} \right\}.$$

(v) Lower range deviation (LRD): It is the most conservative lower range dominated generalized deviation measure. This measure is defined by

$$LRD(X) = \mathbb{E}[X - \inf X].$$

Its acceptance set induced is represented by

$$\mathscr{A}^k_{LRD} = \{X \in L^p : \mathbb{E}[X] - \inf X \leq k\}.$$

(vi) Upper range deviation (URD): Similar to the measure above, however this measure only takes in account the positive part. It is defined conform

$$URD(X) = \mathbb{E}[\sup X - X],$$

and its acceptance set induced is given by

$$\mathscr{A}^k_{URD} = \{X \in L^p : \sup X - \mathbb{E}[X] \leq k\}.$$

(vii) Full range deviation (FRD): A very conservative measure that considers the full range of the position. It can be defined as

$$FRD(X) = \sup X - \inf X,$$

and its acceptance set induced is

$$\mathscr{A}^k_{FRD} = \{X \in L^p : \sup X \leq k + \inf X\}.$$

(viii) Value at risk deviation (*VaRD*): This measure is a deviation measure induced by value at risk (VaR). It is defined, $\forall\, \alpha \in [0,1]$, as

$$VaRD_\alpha(X) = VaR_\alpha(X - \mathbb{E}[X]) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \geq \mathbb{E}[X + x]) < \alpha\}.$$

Its acceptance set induced is given by

$$\mathscr{A}^k_{VaRD_\alpha} = \{X \in L^p : \mathbb{P}(X < \mathbb{E}[X - k]) \leq \alpha\}\,.$$

(ix) Expected shortfall deviation (*ESD*): This measure is a generalized deviation measure induced by the expected shortfall (ES), being defined, $\forall\, \alpha \in [0,1]$, as

$$ESD_\alpha(X) = ES_\alpha(X - \mathbb{E}[X]) = \mathbb{E}[-(X - \mathbb{E}[X]) \mid X \leq -VaR_\alpha(X)].$$

Its acceptance set induced is defined as

$$\mathscr{A}^k_{ESD_\alpha} = \left\{X \in L^p : \mathbb{E}[X|X \geq -VaR_\alpha(X)] \leq \frac{\alpha k}{1-\alpha}\right\}.$$

5 Conclusion

In this work we proposed a class of deviations measures induced by some set. This class have the novel interpretation of being the amount that should be shrunk for the position to be acceptable. We analyzed how properties and operations reflect on the set-induced deviation. As a by-result we have the acceptance sets for deviation measures. Any positive homogeneous deviation measure can be fitted in our approach. When the set that generates the set-induced deviation is an acceptance set of some positive homogeneous deviation measure, the set-induced deviation is a scaled version of it. The acceptance set of the set-induced deviation measure is equal to the set that generate it. It is worth highlighting that our measure only is a proper deviation measure when the set is closed for scalar addition and radially bounded at non-constants. Therefore, the approach of our set-induced deviation has no real need be bounded to deviation measures. It could represent the amount to be shrunk for a position to be acceptable in any financial aspect, be it, risk, liquidity, cost, deviation, or even finitely many altogether. It could be done be taking $\mathscr{A}$ as being a sub-level set of a risk, liquidity, cost or deviation measure or its intersection.

Bibliography

ACERBI, C.; TASCHE, D. On the coherence of expected shortfall. **Journal of Banking & Finance**, Elsevier, v. 26, n. 7, p. 1487–1503, 2002.

ARTZNER, P.; DELBAEN, F.; EBER, J.-M.; HEATH, D. Coherent measures of risk. **Mathematical Finance**, v. 9, n. 3, p. 203–228, 1999.

ARTZNER, P.; DELBAEN, F.; KOCH-MEDINA, P. Risk measures and efficient use of capital. **ASTIN Bulletin: The Journal of the IAA**, v. 39, n. 1, p. 101–116, 2009.

BERKHOUCH, M.; LAKHNATI, G.; RIGHI, M. B. Extended Gini-type measures of risk and variability. **Applied Mathematical Finance**, Routledge, v. 25, n. 3, p. 295–314, 2018.

DELBAEN, F. Coherent risk measures on general probability spaces. Springer, p. 1–37, 2002.

FÖLLMER, H.; SCHIED, A. **Stochastic finance: an introduction in discrete time**. 4. ed. : Walter de Gruyter, 2016.

FRITTELLI, M.; SCANDOLO, G. Risk measures and capital requirements for processes. **Mathematical Finance**, Wiley Online Library, v. 16, n. 4, p. 589–612, 2006.

FURMAN, E.; WANG, R.; ZITIKIS, R. Gini-type measures of risk and variability: Gini shortfall, capital allocations, and heavy-tailed risks. **Journal of Banking & Finance**, v. 83, p. 70–84, 2017.

GRECHUK, B.; MOLYBOHA, A.; ZABARANKIN, M. Maximum entropy principle with general deviation measures. **Mathematics of Operations Research**, INFORMS, v. 34, n. 2, p. 445–467, 2009.

KOCH-MEDINA, P.; MUNARI, C.; ŠIKIĆ, M. A simple characterization of tightness for convex solid sets of positive random variables. **Positivity**, Springer, v. 22, n. 4, p. 1015–1022, 2018.

KROKHMAL, P.; ZABARANKIN, M.; URYASEV, S. Modeling and optimization of risk. **Surveys in Operations Research and Management Science**, v. 16, n. 2, p. 49–66, 2013.

MARKOWITZ, H. Portfolio selection. **The Journal of Finance**, v. 7, n. 1, p. 77–91, 1952.

PFLUG, G. C. Subdifferential representations of risk measures. **Mathematical Programming**, v. 108, n. 2-3, p. 339–354, 2006.

RIGHI, M. B. A composition between risk and deviation measures. **Annals of Operations Research**, Springer, 2018.

RIGHI, M. B.; BORENSTEIN, D. A simulation comparison of risk measures for portfolio optimization. **Finance Research Letters**, Elsevier, v. 24, p. 105–112, 2018.

RIGHI, M. B.; CERETTA, P. S. Shortfall deviation risk: An alternative for risk measurement. **Journal of Risk**, v. 9, p. 81–116, 2016.

ROCKAFELLAR, R. T.; URYASEV, S. The fundamental risk quadrangle in risk management, optimization and statistical estimation. **Surveys in Operations Research and Management Science**, Elsevier, v. 18, n. 1-2, p. 33–53, 2013.

ROCKAFELLAR, R. T.; URYASEV, S.; ZABARANKIN, M. Generalized deviations in risk analysis. **Finance and Stochastics**, Springer, v. 10, n. 1, p. 51–74, 2006.

ROCKAFELLAR, R. T.; URYASEV, S.; ZABARANKIN, M. Master funds in portfolio analysis with general deviation measures. **Journal of Banking & Finance**, Elsevier, v. 30, n. 2, p. 743–778, 2006.

ROCKAFELLAR, R. T.; URYASEV, S.; ZABARANKIN, M. Equilibrium with investors using a diversity of deviation measures. **Journal of Banking & Finance**, Elsevier, v. 31, n. 11, p. 3251–3268, 2007.