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A simple thought experiment to discuss the mass–energy equivalence in the special theory of relativity

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Abstract
Einstein's relation between mass and energy is perhaps the most famous equation of Physics. Despite its simplicity, the meaning of $E_0 = mc^2$ is not easy to grasp. Furthermore, its traditional derivations rely either on the integral of *momentum*, on properties of electromagnetic radiation, or even on the expression for transformation of energy. In the present work, we provide a simple thought experiment with an inelastic collision between two particles observed from two inertial reference frames. We show that for the conservation of relativistic *momentum* to hold, the mass of the system must increase after the collision. We also show that the increase of mass relates to the loss of kinetic energy according to the equation $\Delta K = -\Delta mc^2$, which enables us to define the equation for relativistic energy ($E = mc^2/\sqrt{1 - v^2/c^2}$), rest energy ($E_0 = mc^2$) and relativistic kinetic energy ($K = E - E_0$). There are two main advantages in this presentation: first, it relies only on simple algebra, not depending on differential calculus and on any property of radiation; second, it leads directly to a comprehensible physical meaning of the relation of equivalence, which can sometimes be too obscure in more formal derivations.

Keywords: special theory of relativity, relativity, equivalence relation, mass, energy

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1. Introduction

The relation between mass and energy discussed by Albert Einstein (1992a) in 1905 originated the most famous equation of Physics ($E_0 = mc^2$). Its meaning, however, is not easy to grasp since it contradicts our very intuitive notions about these two concepts. Also, the difficulty in comprehending the relation of equivalence is due to the fact that the equation brings a whole new understanding to the conception of *inertia* that does not fit in Newtonian physics (Jammer 2000). Furthermore, Einstein's relation is also encompassed by a dispute around the concept of 'relativistic mass' (Kneubil 2018b)—which can also bring uncertainty to the meaning of this concept in the context of special theory of relativity[3].

Another aspect that can be challenging in a first presentation of the equivalence relation is its derivation. Albeit mathematically simple, Einstein's original derivation already relied on the expression of energy transformation. Without using differential calculus, Einstein has also presented other derivations, but they still eventually depend on properties of electromagnetic radiation (Einstein 1992c). Moreover, popular contemporary derivations are usually based on defining the expression of work, a method that demands the computation of an integral (Tipler and Llewellyn 2012, p 70). In any case, the derivation can be difficult to follow in introductory levels or it can lead to the wrong impression that the equation is only valid for the case of absorption or emission of radiation.

In the present work, we discuss a simple thought experiment with an inelastic collision between two particles, i.e. a collision in which *momentum* is conserved, but kinetic energy is not. Relying on the expression of relativistic *momentum* and on the velocity-addition formula, we show that, after the collision, the mass of each particle increases. If we suppose that the mass increase (Δm) is proportional to the loss of kinetic energy (ΔK), it is possible to show that $\Delta K = -\Delta mc^2$.

The study of this equation allows us to define three concepts, the total relativistic energy $\left(E = \frac{mc^2}{\sqrt{1-\beta^2}}\right)$, the rest energy $E_0 = mc^2$ and kinetic energy $(K = E - E_0)$.

The first advantage of this derivation is that it relies only on basic principles of the special theory of relativity and basic mathematics. More specifically, along the analysis, we use the following physical concepts:

(a) velocity-addition formula of the Special Theory of Relativity
(b) the relation between relativistic momentum and mass
(c) relativistic momentum conservation
(d) reduction of relativistic expressions into Newtonian expressions in the limit of low speed.

And we need the following mathematical tools:

(a) algebra
(b) Newton's Binomial.

In section 2, we review the basic concepts of STR that are necessary to discuss the thought experiment. In section 3, we present the thought experiment and the derivation of the equations, then discuss their meanings. In section 4, we present our concluding remarks. The mathematical framework used allows this approach to be presented in high school or introductory undergraduate Physics courses.

2. Fundamental concepts of the Special Theory of Relativity

In 1905, Albert Einstein (1992b) showed that, for electromagnetic theory to be considered valid in all inertial reference frames, important alterations should be done in Newtonian Physics. Proposing the principle of relativity (i.e. all laws of Physics must be valid in reference frames in which mechanics holds good) and the principle of constancy of the speed of light (i.e. the speed of light in vacuum is the same independently of the speed

[3] In this paper, we adopt the notation according to which mass is an invariant property (Okun 1989, Hecht 2009). In this case, it is not necessary to make a distinction between m_0 (rest mass) and m (mass). What we distinguish is energy $(E = mc^2/\sqrt{1-v^2/c^2})$ from rest energy $(E_0 = mc^2)$. It should be noted however that the analysis to be presented steers away from this controversy. Independently of using m_0 or m, the derivation and the analysis remain valid.

of the source), Einstein has shown that the set of equations that relate the coordinates in a reference S' moving with velocity $+V$ along the x-axis in relation to a reference frame S is given by the Lorentz Transformations. For the x-component, one has

$$x' = \frac{(x - Vt)}{\sqrt{1 - \frac{V^2}{c^2}}}. \quad (1)$$

Since S' is moving in the x-direction, the other coordinates remain unchanged in both frames (i.e. $y' = y$ and $z' = z$). Also, in special theory of relativity, the time coordinate is different in S' and S:

$$t' = \frac{\left(t - \frac{Vx}{c^2}\right)}{\sqrt{1 - \frac{V^2}{c^2}}}. \quad (2)$$

Neither the spatial coordinate nor the time are measured equally in both systems, thus velocities also assume different values in different inertial frames. For a velocity v measured in the reference S along the x-direction, the value v' measured in S' is

$$v' = \frac{\Delta x'}{\Delta t'} = \frac{(v - V)}{1 - \frac{vV}{c^2}}. \quad (3)$$

An important feature of these new relations is that in the limit of low velocity V between the two frames, they reduce to what one would expect in Newtonian mechanics (i.e. the Galilean transformations).

In this context, it is possible to show that in the same way that in Newtonian Mechanics there is a property called *momentum*, which is always conserved for a closed system, in special theory of relativity, there is a property called relativistic *momentum*, which is also always conserved for a closed system (Tipler and Llewellyn 2012, p 66). The relativistic *momentum* may be written as

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (4)$$

in which m is the mass of the body. The mass in expression (4) is invariant, i.e. it is considered to be independent of the velocity of the body[4]. Furthermore, since the relativistic momentum should be conserved for a closed system, we may say that

$$\sum p = \text{constant}. \quad (5)$$

Thus, for instance, in a collision process, the sum of the relativistic *momentum* of all the particles before the collision must be equal to the sum of the relativistic *momentum* of all the particles after the collision. Although the *momentum* assumes different values in different inertial reference frames, its value is kept constant in each frame. Finally, it should be noted that for low speeds, the relativistic momentum reduces to the Newtonian expression.

3. A simple thought experiment

In a laboratory (inertial reference frame S'), a collision between two particles, each one with mass m, is carried out for an experiment. In this frame, particle 1 has an initial velocity equal to $v_1' = +v$, and particle 2 has an initial velocity equal to $v_2' = -v$. The collision is perfectly inelastic, which means that the two particles stick together in reference S' after the collision.

When an inelastic collision happens, even though the momentum is conserved, the kinetic energy is not. The lost kinetic energy is considered to be dissipated in different forms of energy such as with internal modes of movement and vibration. As will be discussed, this increase of internal energy leads to a corresponding increase in mass. By studying this collision experiment, it is possible to determine not only the expression of relativistic kinetic energy but also the relation between mass and energy.

In order to do so, one may assume that the collision experiment is also observed in a second reference frame S, moving with velocity $V' = -v$ in relation to S'. It means that the reference frame S is moving in relation to S' with the same velocity of the particle 2. One can think about this as if an observer at rest in S was traveling on particle 2. The thought experiment is represented in figure 1.

[4] As we had said, we use a notation for invariant mass. For those who prefer to work with the concept of relativistic mass, what is used in expression (4) is the rest mass, i.e. $p = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}$. So, in both notations, the 'mass' that is part of the expression of relativistic *momentum* is invariant.

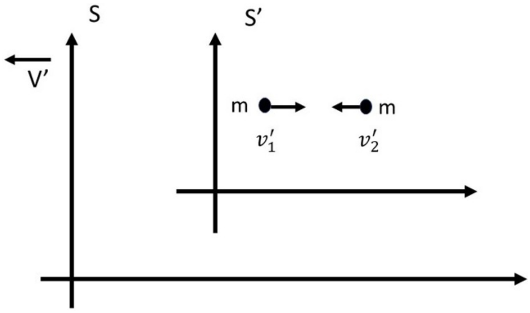

Figure 1. Inelastic collision between two particles observed from two different inertial reference frames.

We start our analysis by describing the conservation of relativistic *momentum* in both inertial reference frames.

3.1. Momentum conservation in S'

In the laboratory frame (S'), the initial momentum (p'_1 and p'_2) of the two particles are

$$p'_1 = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6}$$

and

$$p'_2 = -\frac{mv}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{7}$$

Thus, the total initial momentum of the system is

$$p'_1 + p'_2 = 0. \tag{8}$$

After the collision, the system is bound and at rest, and the total final momentum p_{f} is

$$p'_{\mathrm{f}} = 0. \tag{9}$$

Since the initial total momentum and the final total momentum are the same, the conservation of momentum clearly holds.

3.2. Conservation of momentum in reference frame S

In the reference frame S, before the collision, the velocities of particle 1 and 2 (v_1 and v_2, respectively) can be obtained with the velocity-addition formula (equation (3)):

$$v_1 = \frac{2v}{1+\frac{v^2}{c^2}} \tag{10}$$

and

$$v_2 = 0. \tag{11}$$

The momentum of each particle can be described using equations (4)–(11):

$$p_1 = \frac{2mv}{1+\frac{v^2}{c^2}} \frac{1}{\sqrt{1-\frac{\frac{4v^2}{c^2}}{\left(1+{v^2}/{c^2}\right)^2}}} = \frac{2mv}{1-\frac{v^2}{c^2}} \tag{12}$$

and

$$p_2 = 0. \tag{13}$$

After the inelastic collision, the bound system (that is at rest in S', i.e. $v' = 0$) has the velocity in S given by the velocity addition formula (equation (3)):

$$v_{\mathrm{f}} = \frac{v+0}{1-0.\frac{v}{c^2}} = v. \tag{14}$$

Consequently, considering that the bound system has a mass M, and using the definition of momentum (equation 4):

$$p_{\mathrm{f}} = \frac{Mv_{\mathrm{f}}}{\sqrt{1-\frac{v'^2_{\mathrm{f}}}{c^2}}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{15}$$

Considering the conservation of momentum, the initial total momentum must be equal to the final momentum:

$$\frac{2mv}{1-\frac{v^2}{c^2}} = \frac{Mv}{\sqrt{1-\frac{v^2}{c^2}}} \tag{16}$$

thus, the mass M of the bound system in the reference frame S is

$$M = \frac{2m}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{17}$$

According to equation (17), the mass of the bound system after the collision is greater than the sum of the original masses. Our next step is to investigate how the increase of mass can be related to other physical parameters and how it can be explained.

3.3. Relation between mass and energy

As we have discussed, the mass in the expression of momentum is invariant, so it is the same in the reference frames S and S':

$$M' = M = \frac{2m}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{18}$$

We stress the fact that M is an invariant property, in a way to highlight that equation (18) is not a transformation equation of mass for different inertial frames. It expresses the new mass of the system after the full inelastic collision with the established conditions. Equation (18) leads us to recognize that the mass increase in each particle is

$$\Delta m = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) m. \tag{19}$$

Something in the process of collision is responsible for this increase. Since we are working with an idealized model, the problem does not involve many parameters. In simple lines, we know that the momentum is constant, the mass increased and the kinetic energy diminished. It is reasonable to suppose that the increase of mass in each particle is proportional to their loss of kinetic energy. This hypothesis is from the very beginning, leading us to interpret that the original kinetic energy, that is dissipated in the system through the inelastic collision, is responsible for the increase in mass. This loss of kinetic energy is very clear in the reference frame S, because, after the collision, the particles are at rest (which may not be so obvious in S'). Thus, we can write the proportionality equation in reference frame S:

$$\Delta K = -\alpha \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) m. \tag{20}$$

We have not determined yet the expression for the relativistic kinetic energy. We know, however, that in the low-speed limit, the expression for kinetic energy must reduce to the Newtonian expression. Thus, in the limit of low velocities, the left side of equation should be considered to be

$$\Delta K \approx 0 - \frac{1}{2}mv^2 \tag{21}$$

and the right side of equation (20) can be rewritten using Newton's binomial

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{22}$$

Using equation (22) to determine the expression for $(1+x)^n$ and if $x \approx 0$, one may eliminate all terms with x^k for $k \geqslant 2$. Thus[5],

$$(1+x)^n \approx 1 + nx. \tag{23}$$

Taking $x = \frac{v^2}{c^2}$, then:

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{\mathrm{v}^2}{\mathrm{c}^2}. \tag{24}$$

Substituting the results obtained in (21) and (24) in equation (20):

$$\frac{1}{2}mv^2 = \alpha\frac{1}{2}m\frac{v^2}{c^2}. \tag{25}$$

Thus,

$$\alpha = c^2. \tag{26}$$

It is possible to conclude, therefore, that the loss of energy ΔK was accompanied by an increase of mass Δm that looks like this:

$$\Delta K = -\Delta mc^2. \tag{27}$$

Equation (27) expresses the fact that the dissipation of energy ΔK in internal modes of translational or vibrational movement is responsible for a correspondent increase in mass equal to $\Delta K/c^2$. Also, substituting (19) in (27) and recognizing that $\Delta K = 0 - K_i = -K_i$:

$$\mathrm{K_i} = \left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right) mc^2. \tag{28}$$

[5] Although equation (23) is derived as long as n is a positive integer, the expression is valid for any real n.

Equation (24) is a subtraction of two terms, one that is dependent on the velocity of the body, and the other that is not velocity-dependent. If velocity-independent term E_0 (rest energy) and velocity-dependent-term E (relativistic energy) are brought to the equation, then kinetic energy (the energy related to the movement of a body) is simply the subtraction of a body's total energy and its rest energy. In synthesis, total energy equals

$$E\,(\text{total energy}) = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{29}$$

The rest energy equals

$$E_0\,(\text{rest energy}) = mc^2. \tag{30}$$

And, finally, the kinetic energy looks like this

$$K\,(\text{Kinetic Energy}) = E - E_0. \tag{31}$$

Following our interpretation on equation (27), equation (30) leads us to interpret that the mass of a system is a measurement of its energy content, i.e. what is measured as mass is also taking into account different sorts of internal energy of a body (binding energy, kinetic energy, vibrational energy of the components of the body). Nowadays, there are many physical, concrete examples in which this relation can be used (Kneubil 2018a).

4. Concluding remarks

In this paper, we presented a very simple experiment to derive the famous equation of equivalence between mass and energy ($\boldsymbol{E_0 = mc^2}$). In our proposal, one relies on the concept of conservation of relativistic *momentum* and on the velocity-addition formula. Along the derivation, we only use simple algebra. Besides its simplicity, an important feature of this thought experiment is that it deals with the equivalence between kinetic energy and mass, and not with the equivalence between energy of radiation and mass of a body (as in many other derivations). This specificity is broad enough to allow for a discussion that, according to the special theory of relativity, energy (in different forms) may contribute to the inertia of a physical system.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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