Geometric Nonlinear Dynamic Analysis of Plates and Shells Using Eight-Node Hexahedral Finite Elements with Reduced Integration

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Abstract

This work presents a geometric nonlinear dynamic analysis of plates and shells using eight-node hexahedral isoparametric elements. The main features of the present formulation are: (a) the element matrices are obtained using reduced integrations with hourglass control; (b) an explicit Taylor-Galerkin scheme is used to carry out the dynamic analysis, solving the corresponding equations of motion in terms of velocity components; (c) the Truesdell stress rate tensor is used; (d) the vector processor facilities existing in modern supercomputers were used. The results obtained are comparable with previous solutions in terms of accuracy and computational performance.

Keywords: Plates and Shells, Dynamic Analysis, Geometric Nonlinearity, Finite Elements.

Introduction

Plates and shells are particular cases of three-dimensional solids and can be analyzed by the Finite Element Method (FEM) using different formulations.

Plates analysis can be formulated using Kirchoff hypothesis (1850), which is applied to thin plates, or based on Reissner-Mindlin theory (1945-1951), which includes transverse shear deformation and is mainly applied to thick plates. In spite of the intense development of many specific element formulations for plate bending, based in Kirchoff or Reissner-Mindlin theories, plates in the present work are considered as a special case of shells.

Membrane and bending shell theories were formulated by Lamé and Clapeyron (1833) and Aron (1874). Love (1927) presented the general bases of the shell theory and was followed later on by many authors such as Flügge (1934), Green and Zerna (1954), Novozilhov (1959), Timoshenko and Woinowsky-Krieger (1959), Flügge (1960) and Gol'denveizer (1961). Closed form analytical solutions were found for simple cases. In the last four decades, since the introduction of the FEM, this technique was applied to obtain solutions for more complex problems.

The main type of elements employed in the analysis of shells are: the plane element (approximating a curved surface by an assembling of flat elements), the curved elements based on shell theories and the curved elements obtained by degeneration of three-dimensional solid elements.

When shells are analyzed using flat elements several problems appear and, sometimes, they can lead to wrong solutions. The main problems are: coupling between bending and membrane behavior, difficulties to handle with inter-element boundaries in coplanar elements and the existence of spurious bending moments in inter-element boundaries. In order to improve shell representation many authors have used curved elements based on some shell theory and employing curvilinear coordinates. However, some difficulties arise in the generalized strain-displacement definition, in selecting a theory in which the strain energy is equal to zero for rigid body displacements and in the correct definition of angular displacements.

Three-dimensional solid elements have been also used to model shells and, for this case, a particular shell theory is not necessary. In order to reduce the number of degrees of freedom and to avoid numerical drawbacks arising when three-dimensional solid elements are used, special elements obtained as degeneration of these three-dimensional elements were implemented and applied, first by Ahmad et al. (1970) and later by many authors such as Bathe (1996), Hughes (1987) and Liao and Reddy (1987), including different alternatives and improvements. In these type of elements the zero normal stress condition is imposed together with the hypothesis that 'normals' to the middle surface remain straight (but not necessarily 'normals') after the deformation; nodes and corresponding nodal unknowns are defined at the middle surface.

When normal quadrature rules are used in three-dimensional degenerated elements, they tend to 'lock' in thin shells applications, especially for low-order elements. On the other hand, if a selective reduced integration rule is used to prevent shear locking, good results can be obtained for thin shells, but in this case rank deficiency (spurious

'mechanisms') may appear and, although sometimes these spurious modes can be precluded from globally forming by appropriated boundary conditions, they represent a potential deficiency. The situation is even more acute when uniform reduced integration rules are used (Hughes, 1987). Another source of problems is the membrane locking. Much research has been undertaken to overcome shear and membrane locking in plate and shells. Many elements have been implemented and behave well for thin shell analysis. It is worthwhile to mention some elements that have been successfully applied such as the MITC family formulated by Bathe and co-workers (1996), or the shell elements implemented by Mc Neal (1978), Huang and Hinton (1986), Park and Stanley (1986), Hughes and Liu (1981), Belytschko, Liu, Ong and Lam (1985) and Belytschko, Stolarsky ,Liu, Carpenter and Ong (1985), among others.

However, in large scale finite element analysis, with many unknowns involved, the efficiency is of crucial importance to reduce computational costs and speed-up the design procedure. The most efficient elements are those with linear interpolation functions and one-point quadrature with hourglass control. One of the first works in this direction was presented by Kosloff and Frazier (1978), but in their formulation it is necessary to solve 4 systems of 8 equations for distorted three-dimensional elements, and this is not cost efficient for dynamic analysis. Flanagan and Belytschko (1981), Belytschko (1983), and Belytschko, Ong, Liu and Kennedy (1984) presented a systematic and effective way to hourglass control, but in both formulations a parameter, to be defined by the user, is required.

Some shell elements with one-point quadrature were formulated by Belytschko, Lin and Tsay (1984), Hallquist, Benson and Goudreau (1986), Liu, Law, Lam and Belytschko (1986), Belytschko, Wong and Chiang (1992). Belytschko and Binderman (1993) implemented the hourglass control of the eight-node hexahedral element, where the stabilization parameter is not required, although the stabilization matrix still depends on the Poisson coefficient; the aspect was eliminated by Liu, Hu and Belytschko (1994). More recently Zhu and Zacharia (1996) and Key and Hoff (1995) have also presented quadrilateral shell elements with the one-point quadrature and the hourglass control concept.

The geometric nonlinear dynamic analysis of plates and shells using underintegrated eight-node hexahedral elements with hourglass control is presented in this work. Time integration is carried out using an explicit Taylor-Galerkin scheme and the Truesdell stress rate tensor is employed for the nonlinear analysis. Advantages arising from vector processors are also used. Comparative examples show the effectiveness of this 3-D solid element to analyze plates and shells without shear and membrane locking.

The Explicit Taylor-Galerkin Scheme

The equations of motion are given by the following expression (Hughes, 1987):

$$\frac{\partial}{\partial t}\left(\rho\, v_i\right) - \frac{\partial}{\partial x_j}\,\sigma_{ij} + \left[\frac{\mu}{\rho}\left(\rho\, v_i\right) - f_i\right] = 0 \quad (i,j = 1,2,3)\, in \quad \Omega \tag{1}$$

where v_i are the velocity components, ρ is the specific mass, σ_{ij} are the stress tensor components, f_i are the components of the body forces per unit volume and μ is a damping coefficient; t and x_i are the time and spatial coordinates respectively and Ω is the domain.

The boundary conditions are given by:

$$v_i = \bar{v}_i \quad (i = 1,2,3) \; in \; \Gamma_v ; \quad \sigma \; \sigma_{ij} n_j = \bar{P}_i \; (i,j = 1,2,3) \; in \; \Gamma_\sigma \quad (2)$$

where $\bar{v}_i$ are prescribed values of the velocity components, $\bar{P}_i$ are prescribed values of the surface forces at the part Γ_v of the boundary surface, n_j is the cosine of the angle formed by the outward normal to the boundary surface Γ_σ and the coordinate axis x_j; $\Gamma_\sigma \cup \Gamma_v = \Gamma$, being Γ the total boundary surface.

The initial conditions are:

$$u_i = u_i^0 \quad (i = 1,2,3) \; in \; \Omega; \; v_i = v_i^0 \quad (i = 1,2,3) \; in \; \Omega \quad (3)$$

where u_i^0 and v_i^0 are the initial values of the displacement and velocity components, respectively.

Expression (1) can be also written, in compact form, as follows:

$$\frac{\partial \rho v}{\partial t} - \frac{\partial}{\partial x_j} S_j + Q = 0 \quad (j = 1,2,3) \; in \; \Omega \quad (4)$$

with

$$\rho v^T = \left\{ \rho v_1, \rho v_2, \rho v_3 \right\}$$
$$S_j^T = \left\{ \sigma_{1j}, \sigma_{2j}, \sigma_{3j} \right\}$$
$$Q^T = \left\{ \frac{\mu}{\rho}(\rho v_1) - f_1, \frac{\mu}{\rho}(\rho v_2) - f_2, \frac{\mu}{\rho}(\rho v_3) - f_3 \right\} \quad (5)$$

where superposed index T indicates transpose vectors.

Expanding ρ **v** in a Taylor series up to second order terms we obtain:

$$\Delta(\rho v)^{n+1} = (\rho v)^{n+1} - (\rho v)^n = \Delta t \left\{ \frac{\partial}{\partial t} \left[(\rho v)^n + \frac{\Delta t}{2} \frac{\partial}{\partial t} (\rho v)^n \right] \right\} \quad (6)$$

where indexes n+1 and n correspond to time level $t = (n+1)*\Delta t$ and $t = (n+1/2)*\Delta t$, respectively, with Δt being the time step size.

Taking into account expression (4), Eq. (6) may be written in the following form:

$$\Delta(\rho v)^{n+1} = \Delta t \left\{ \frac{\partial}{\partial x_j}\left(S_j\right)^n - Q^n + \frac{\Delta t}{2}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial x_j}\left(S_j\right)^n - Q^n\right]\right\} =$$

$$= \Delta t \left\{ \frac{\partial}{\partial x_j}\left[\left(S_j\right)^n + \frac{\Delta t}{2}\frac{\partial}{\partial t}\left(S_j\right)^n\right] - \left[Q^n + \frac{\Delta t}{2}\frac{\partial}{\partial t}Q^n\right]\right\} =$$

$$= \Delta t \left\{ \frac{\partial}{\partial x_j}\left(S_j\right)^{n+1/2} - \left[\frac{\mu}{\rho}(\rho v)^{n+1/2} - f^{n+1/2}\right]\right\} \qquad (7)$$

with

$$\frac{\mu}{\rho}(\rho v)^{n+1/2} = \frac{\mu}{\rho}\left[(\rho v)^n + \frac{1}{2}\Delta(\rho v)^{n+1}\right]; \quad f^{n+1/2} = \frac{1}{2}\left(f^{n+1} + f^n\right) \qquad (8)$$

In expressions (7) and (8), superposed index n+1/2 corresponds to time level $t = (n+1/2)*\Delta t$.

Introducing the damping term of expression (8) in (7) and taking into account that $(\mathbf{S_j})^{n+1/2}$ is a function of $\mathbf{u}^{n+1/2}$ we can write:

$$\left(1 + \frac{\mu}{2\rho}\Delta t\right)\Delta(\rho v)^{n+1} = \Delta t \left\{\frac{\partial}{\partial x_j}S_j\left(u^{n+1/2}\right) - \frac{\mu}{\rho}(\rho v)^n + f^{n+1/2}\right\} \qquad (9)$$

Applying the classical Galerkin weighted residual method, the following matrix expression is obtained at element level:

$$\beta M_C \Delta(\rho \bar{v})^{n+1} = \Delta t \left\{F_{int}\left(\bar{u}\right)^{n+1/2} - \frac{\mu}{\rho}M_C(\rho \bar{v})^n + P_f^{n+1/2} + P_p^{n+1/2}\right\} = \Delta t\, H \qquad (10)$$

where

$$M_C = \int_{\Omega e} N^T N d\Omega; \quad P_f = \int_{\Omega e} N^T f^{n+1/2} d\Omega; \quad P_p = \int_{\Gamma e} N^T \bar{p}^{n+1/2} d\Gamma$$

$$F_{int}\left(\bar{u}^{n+1/2}\right) = \int_{\Omega e} B^T \sigma^{n+1/2} d\Omega; \quad \beta = 1 + \frac{\mu}{\rho}\frac{\Delta t}{2} \qquad (11)$$

In expression (11), **N** is a vector containing the element shape functions, **B** is a matrix involving shape functions derivatives, $\mathbf{M_C}$ is the consistent mass matrix and $F_{int}\left(\bar{u}^{n+1/2}\right)$ is the vector of internal forces at the time level $t = (n+1/2)*\Delta t$.

$$\bar{u}^{n+1/2} = \bar{u}^n + \frac{\Delta t}{2}\frac{\partial \bar{u}^n}{\partial t} = \bar{u}^n + \frac{\Delta t}{2}\bar{v}^n \qquad (12)$$

In order to get $F_{int}\left(\bar{u}^{n+1/2}\right)$, it is necessary to compute $\bar{u}^{n+1/2}$ and this can be made using the following expression:

Equation (10) is modified to obtain the final recurrence expression, wich is given by:

$$\Delta(\rho\bar{v})_{k+1}^{n+1} = \frac{\Delta t}{\beta} M_L^{-1} H + \beta M_L^{-1}(M_L - M_C)\Delta(\rho\bar{v})_k^{n+1} \qquad (13)$$

where M_L is the lumped mass matrix and the index k is an iteration counter. The term $\Delta(\rho\bar{v})_k^{n+1}$ changes in each iterative step, until convergence is achieved, but all other terms in the right hand side of Eq. (13) remain constant during the iterative process. In practical applications no significant differences were obtained in the results when these iterations were omitted.

After assembly, applying boundary conditions and solving (13), velocity and displacement components are calculated with the following expressions:

$$\bar{v}^{n+1} = \frac{1}{\rho}\left[(\rho\bar{v})^n + \Delta(\rho\bar{v})_{k+1}^{n+1}\right]; \bar{u}^{n+1} = \bar{u}^n + \frac{\Delta t}{2}\left(\bar{v}^{n+1} + \bar{v}^n\right) \qquad (14)$$

In order to preserve numerical stability, it is necessary to take $\Delta t \leq \Delta t_{crit}$, where Δt_{crit} is the critical time step determined by the Courant-Friedrichs-Levy (CFL) stability condition (Richtmyer and Morton, 1967).

The computational procedure is given by the following steps:

a. Calculate $t = (n+1)\Delta t$;
b. Calculate $\Delta\bar{u}^{n+1/2} = \bar{u}^{n+1/2} - \bar{u}^n = \Delta t/2 \bar{v}^n$;
c. Calculate $\Delta\varepsilon^{n+1/2}, \Delta\sigma^{n+1/2}$ and $\Delta F_{int}^{n+1/2}$;
d. Calculate $F_{int}^{n+1/2} = F_{int}^n + \Delta F_{int}^{n+1/2}$;
e. Calculate $\Delta(\rho v)_{k+1}^{n+1}$ with (13);
f. Calculate $\bar{v}^{n+1}$ and $\bar{u}^{n+1}$ with (14);
g. Calculate $\Delta\varepsilon^{n+1}$ with $\Delta\bar{u}^{n+1} = \bar{u}^{n+1} - \bar{u}^{n+1/2}$, $\Delta\sigma^{n+1} = T(\sigma)\Delta\varepsilon^{n+1}$, where $T(\sigma)$
h. is the constitutive matrix.
i. Calculate ΔF_{int}^{n+1} with $\Delta\sigma^{n+1}$ obtained in the previous step;
j. Calculate $\varepsilon^{n+1} = \varepsilon^n + \Delta\varepsilon^{n+1}, \sigma^{n+1} = \sigma^n + \Delta\sigma^{n+1}$ and $F_{int}^{n+1} = F_{int}^{n+1/2} + \Delta F_{int}^{n+1}$;
k. If $t < t_{total}$ then return to (a), otherwise go to (k);
l. End of the process.

The underintegrated eight-node hexahedrical element with hourglass control

For an eight-node hexahedrical isoparametric element, the shape functions are given by:

$$\mathbf{N_n}(\xi, \eta, \zeta) = 1/8(1 + \xi_{n\xi})(1 + \eta_{n\eta})(1 + \zeta_{n\zeta}) \quad (n = 1, .., 8) \quad (15)$$

where ξ_n, η_n and ζ_n are the natural coordinates ξ, η and ζ of the element node n in consideration.

The eight-node hexahedrical element is indicated in <u>Fig. 1</u> with respect to its local system of reference ξ, η and ζ and with respect to its global system of reference x_1, x_2 and x_3.
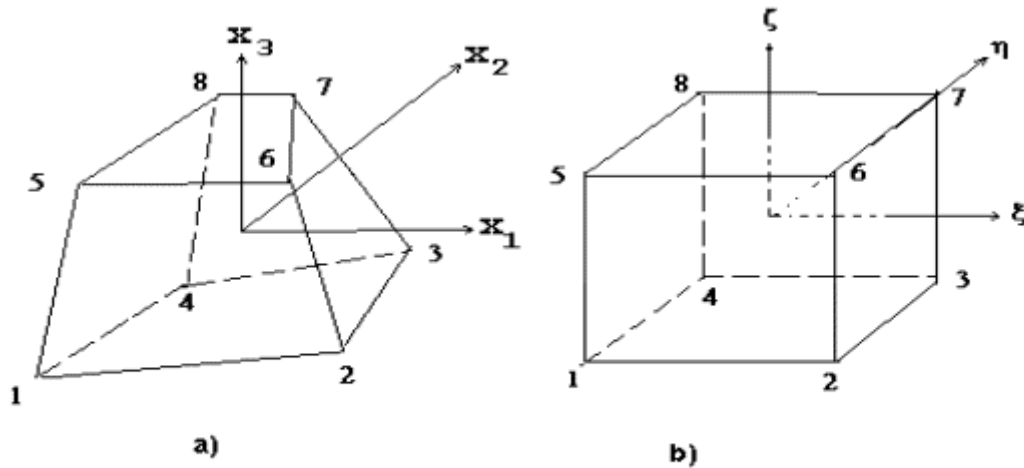

Fig. 1 Hexahedrical isoparametric element: (a) referred to the global system (x_1, x_2, x_3); (b) referred to the element system (ξ, η, ζ)

The Jacobian matrix at the element center is given by:

$$J(0) = \frac{1}{8} \begin{bmatrix} \xi^T x_1 & \xi^T x_2 & \xi^T x_3 \\ \eta^T x_1 & \eta^T x_2 & \eta^T x_3 \\ \zeta^T x_1 & \zeta^T x_2 & \zeta^T x_3 \end{bmatrix} \tag{16}$$

where ξ, η and ζ are vectors containing nodal coordinates with respect to the referential system ξ, η and ζ and x_1, x_2 and x_3 are the nodal coordinates with respect to the global system x_1, x_2 and x_3. It can be shown that $|J(0)| = \Omega/8$, where $|J(0)|$ is the determinant of $J(0)$.

The matrix $\mathbf{B_n}(0)$, which contains the shape functions derivatives for each node n at the center of the element is given by:

$$B_n(0) = \begin{bmatrix} N_{n,x_1}(0) \\ N_{n,x_2}(0) \\ N_{n,x_3}(0) \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_{1n} \\ b_{2n} \\ b_{3n} \end{bmatrix} \tag{17}$$

where $\mathbf{N_{n,xi}}(0) = \partial\,\mathbf{N_n}(0)/\partial\,\mathbf{x_i}$ for i=1,2,3.

If $\mathbf{J}(0)^{-1}$ is denoted by $\mathbf{G}$, the following expressions for vectors $\mathbf{b_1}, \mathbf{b_2}$ and $\mathbf{b_3}$ are obtained:

$$b_1 = \frac{1}{8}\{G_{11}\xi + G_{12}\eta + G_{13}\varsigma\}\,;\, b_2 = \frac{1}{8}\{G_{21}\xi + G_{22}\eta + G_{23}\varsigma\}\,;\, b_3 = \frac{1}{8}\{G_{31}\xi + G_{32}\eta + G_{33}\varsigma\} \quad (18)$$

To identify the spurious mode patterns (or zero energy modes or 'hourglass' modes for the hexahedron) resulting from the non-constant strain field, due to the use of one-point quadrature, the following vectors are defined (Flanagan and Belytschko, 1981):

$$h_1^T = [+\,1,-1,+1,-1,+1,-1,+1,-1]\,;\, h_2^T = [+\,1,-1,-1,+1,-1,+1,+1,-1]\,;$$
$$h_3^T = [+\,1,+1,-1,-1,-1,-1,+1,+1]\,;\, h_4^T = [-\,1,+1,-1,+1,+1,-1,+1,-1] \quad (19)$$

In Figure 2 a sketch of bending, torsion and non-physical displacement modes associated to these vectors are shown (Koh and Kikuchi, 1987).
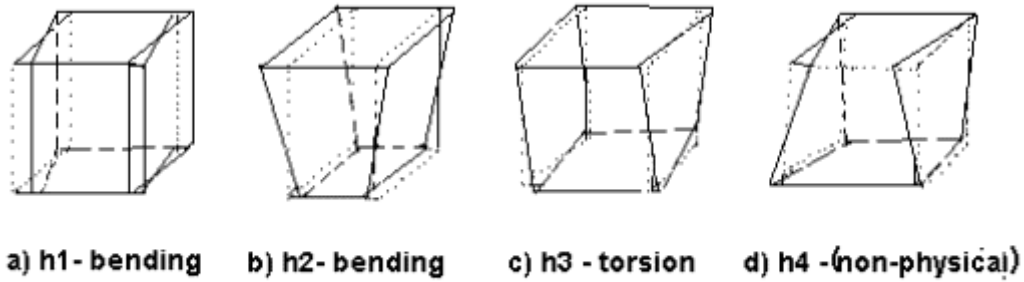

a) h1- bending   b) h2- bending   c) h3 - torsion   d) h4 -(non-physical)

Fig. 2   Spurious mode patterns

Strain and displacement components are related by the expression:

$$\varepsilon = \sum_{n=1}^{8} \hat{B}_n(\xi,\eta,\varsigma)u_n \quad (20)$$

where $\hat{B}_n$ contains the shape functions derivatives for each element node 'n' evaluated at the integration points and u_n is a vector which contains displacement components of the element node 'n'.

Expanding ε in a Taylor series about the element center up to bilinear terms and taking into account expression (20), it is obtained:

$$\hat{B}_n(\xi\,\eta\,\zeta) = B_n(0) + B_{n,\xi}(0)\xi + B_{n,\eta}(0)\eta + B_{n,\zeta}(0)\zeta + 2B_{n,\xi\eta}(0)\xi\eta + 2B_{n,\eta\zeta}(0)\eta\zeta + 2B_{n,\zeta\xi}(0)\zeta\xi$$
$$(n = 1,\ldots 8)$$

(21)

The strain operator $\hat{B}(\xi,\eta,\zeta)$ may be decomposed in its dilatational and deviatoric part. To avoid volumetric locking the dilatational part of the strain operator, $\hat{B}^{dil}(\xi\,\eta\,\zeta)$ is evaluated with a one-point quadrature and consequently all linear and bilinear terms disappear for this part, but remaining for the deviatoric strain operator $\hat{B}^{dev}(\xi\,\eta\,\zeta)$. Then, expression (21) may be written as follows:

$$\hat{B}_n(\xi\,\eta\,\zeta) = \hat{B}_n^{dil}(\xi\,\eta\,\zeta) + \hat{B}_n^{dev}(\xi\,\eta\,\zeta) = B_n(0) + B_{n,\xi}^{dev}(0)\xi + B_{n,\eta}^{dev}(0)\eta + B_{n,\zeta}^{dev}(0)\zeta +$$
$$2B_{n,\xi\eta}^{dev}(0)\xi\eta + 2B_{n,\eta\zeta}^{dev}(0)\eta\zeta + 2B_{n,\zeta\xi}^{dev}(0)\zeta\xi \quad (n = 1,\ldots 8)$$

(22)

To eliminate shear locking, the deviatoric strain sub-matrices can be written in an orthogonal co-rotational coordinate system, rotating with the element. Only one linear term is left for shear strain components and thus removing modes causing shear locking.

In the co-rotational coordinate system components of the strain submatrices, after removing volumetric and shear locking, can be written:

$$\hat{B}_{xx}(\xi\,\eta\,\zeta) = B_{xx}(0) + B_{xx,\xi}^{dev}(0)\xi + B_{xx,\eta}^{dev}(0)\eta + B_{xx,\zeta}^{dev}(0)\zeta + 2\,B_{xx,\xi\eta}^{dev}(0)\xi\eta$$
$$+ 2\,B_{xx,\eta\zeta}^{dev}(0)\eta\zeta + 2\,B_{xx,\zeta\xi}^{dev}(0)\zeta\xi$$

(23.a)

with similar expressions for $\hat{B}_{vv}(\xi,\eta,\zeta)$ and $\hat{B}_{zz}(\xi,\eta,\zeta)$.

$$\hat{B}_{xy}(\xi\,\eta\,\zeta) = B_{xy}(0) + B_{xy,\zeta}^{dev}(0)\zeta \quad \text{(23.b)}$$

$$\hat{B}_{yz}(\xi\,\eta\,\zeta) = B_{yz}(0) + B_{yz,\xi}^{dev}(0)\xi \quad \text{(23.c)}$$

$$\hat{B}_{zx}(\xi\,\eta\,\zeta) = B_{zx}(0) + B_{zx,\eta}^{dev}(0)\eta \quad \text{(23.d)}$$

As was observed by Belytschko and Bindeman (1993), in order to get that skewed elements (evaluated with one-point quadrature) pass the patch test, it is necessary to substitute b_{1n}, b_{2n} and b_{3n}, in expression (17), by the uniform gradient matrices b_{1n}, b_{2n} and b_{3n}, defined by Belytschko and Flanagan (1981), and given by:

$$\bar{b}_{in} = \frac{1}{\Omega_e}\int_{\Omega_e} b_{in}(\xi\,\eta\,\zeta)d\Omega = \frac{1}{\Omega_e}\int_{\Omega_e}\frac{\partial N_n}{\partial x_i}(\xi\,\eta\,\zeta)d\Omega \quad (i = 1,2,3; n = 1,\ldots,8)$$

(24)

Gradient vectors b_1, b_2 and b_3 satisfy the following relations:

$$\bar{b}_i^T x_j = \delta_{ij}; \quad \sum_{n=1}^{8}\bar{b}_{in} = 0; \quad \bar{b}_i^T h_4 = 0 \ (i,j = 1,2,3)$$

(25.a)

$$\bar{b}_i^T h_j = \begin{cases} = 0 \ \text{for non distorted elements} \\ \neq 0 \ \text{for skewed elements} \end{cases} \quad (i,j = 1,2,3) \quad (25.b)$$

where δ_{ij} is the Kronecker delta.

The final form of matrix B in the co-rotational system, obtained after some algebric work and taken into account Eqs. (16) to (24), is given by the following expression:

$$\bar{B} = \begin{bmatrix} \bar{B}_{xx}(\xi\eta\varsigma) \\ \bar{B}_{yy}(\xi\eta\varsigma) \\ \bar{B}_{zz}(\xi\eta\varsigma) \\ \bar{B}_{xy}(\xi\eta\varsigma) \\ \bar{B}_{yz}(\xi\eta\varsigma) \\ \bar{B}_{zx}(\xi\eta\varsigma) \end{bmatrix} = \bar{B}(0) + \bar{B}^{HC}(\xi\eta\varsigma) = \begin{bmatrix} \bar{b}_1^T & 0 & 0 \\ 0 & \bar{b}_2^T & 0 \\ 0 & 0 & \bar{b}_3^T \\ \bar{b}_2^T & \bar{b}_1^T & 0 \\ 0 & \bar{b}_3^T & \bar{b}_2^T \\ \bar{b}_3^T & 0 & \bar{b}_1^T \end{bmatrix} + \frac{1}{8} \begin{bmatrix} \frac{2}{3}f_1 & -\frac{1}{3}f_2 & -\frac{1}{3}f_3 \\ -\frac{1}{3}f_1 & \frac{2}{3}f_2 & -\frac{1}{3}f_3 \\ -\frac{1}{3}f_1 & -\frac{1}{3}f_2 & \frac{2}{3}f_3 \\ G_{22}\varsigma\gamma_3^T & G_{11}\varsigma\gamma_2^T & 0 \\ 0 & G_{33}\xi\gamma_2^T & G_{11}\xi\gamma_1^T \\ G_{33}\eta\gamma_3^T & 0 & G_{11}\eta\gamma_1^T \end{bmatrix} \quad (26)$$

where

$$f_1 = G_{11}(\eta\gamma_1^T + \varsigma\gamma_2^T + 2\eta\varsigma\gamma_4^T); \quad f_2 = G_{22}(\xi\gamma_1^T + \varsigma\gamma_3^T + 2\xi\varsigma\gamma_4^T)$$
$$f_3 = G_{33}(\xi\gamma_2^T + \eta\gamma_3^T + 2\xi\eta\gamma_4^T); \quad G_{ii} = \left[J^{-1}(0)\right]_{ii} \quad (i = 1,2,3)$$
$$\gamma_i = h_i - \left(h_i^T x_i\right)\bar{b}_i \quad (i = 1,2,3,4)$$

In Eq. (26) $\bar{B}(0)$ is a matrix formed with vectors $\bar{b}_1, \bar{b}_2$ and $\bar{b}_3$, computed as indicated in Eq. (24), and $\bar{B}^{HC}(\xi\eta\varsigma)$ is the stabilization matrix , which controls the hourglass modes. These matrices are evaluated using 4 integration points with coordinates (a,a,a), (-a,-a,a), (-a,a,-a) and (a,-a,-a), where $a = \sqrt{3}/3$. γ_1, γ_2, γ_3 and γ_4 are stabilization vectors. Use of 4 integration points is very important in problems where a plastic front exists, because it is difficult to define accurately such plastic front with only one integration point.

Assuming that the determinant of the Jacobian matrix is constant and equal to 1/8 of the element volume, the internal force vector, expressed in (11), may be evaluated by the following expression:

$$F_{int} = \int_{\Omega_e} \bar{B}^T \sigma d\Omega = \frac{\Omega_e}{4} \sum_{i=1}^{4} \bar{B}^T(\xi_i,\eta_i,\varsigma_i) \ \sigma(\xi_i,\eta_i,\varsigma_i) \quad (27)$$

Expression (27) may be rearranged to a more convenient form and can be written as,

$$F_{int} = F_{int}^{ip} + F_{int}^{stab} = \frac{\Omega_e}{4} \sum_{i=1}^{4} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ & \sigma_{22} & \sigma_{23} \\ sym. & & \sigma_{33} \end{bmatrix} \begin{Bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{Bmatrix}_{(\xi_i, \eta_i, \zeta_i)}$$

$$+ \frac{\Omega_e}{4} \sum_{i=1}^{4} \begin{bmatrix} \sigma_{11}^{dev} f_1^T + r_3(\sigma_{12}G_{22}\zeta + \sigma_{13}G_{33}\eta) \\ \sigma_{22}^{dev} f_2^T + r_2(\sigma_{21}G_{11}\zeta + \sigma_{23}G_{33}\xi) \\ \sigma_{33}^{dev} f_3^T + r_1(\sigma_{32}G_{22}\xi + \sigma_{31}G_{1}\eta) \end{bmatrix}_{(\xi_i, \eta_i, \zeta_i)} \quad (28)$$

where F_{int}^{ip} is the internal force vector evaluated at the four integration points and F_{int}^{stab} is the stabilization vector.

The elimination of shear locking in plate or shell elements depends on the proper treatment of the shear strain. To this end it is necessary to use a co-rotational coordinate system attached to the element and rotating together with this element. Assume that x_1, x_2 and x_3 are the coordinate axes of the global system and $\hat{x}_1, \hat{x}_2$ and $\hat{x}_3$ the corresponding coordinates axes in the co-rotational system (these axes will be coincident with the local system ξ, η and ζ for undistorted elements). Two vectors r_1 and r_2, coincident with ξ and η for undistorted elements, are defined as follows:

$$r_{1_i} = \xi^T . x_i; \quad r_{2_i} = \eta^T . x_i \quad (i = 1, 2, 3) \quad (29)$$

A correction term r_c is added to r_2 such that

$$r_1^T . (r_2 + r_c) = 0 \quad \Rightarrow \quad r_c = -\frac{r_1^T . r_2}{r_1^T . r_1} r_1 \quad (30)$$

The orthogonal system is completed with

$$r_3 = r_1 \otimes (r_2 + r_c) \quad (31)$$

Normalizing vectors r_1, $r_2 + r_c$ and r_3, the elements of the rotation matrix R are obtained. The components of R are given by:

$$R_{1i} = \frac{r_{1i}}{|r_1|}; \quad R_{2i} = \frac{r_{2i} + r_{ci}}{|r_2 + r_c|}; \quad R_{3i} = \frac{r_{3i}}{|r_3|} \quad (i = 1, 2, 3) \quad (32)$$

The coordinates may be transformed from the global system x to the co-rotational system $\hat{x}$ by $\hat{x} = Rx$, where R is a matrix of order 24x24 with sub-matrices R of order 3x3 in its diagonal blocks and 0 in all the off-diagonal blocks. A similar expression is used to transform velocity components. The vector F_{int}, given in (28), is expressed in the co-rotational system and the transformation to the global system is carried out multiplying R^1 by the internal force vector referred to the co-rotational system.

Constitutive Equations

The Truesdell stress rate tensor (Prager, 1961) will be used in this work. The components of the Cauchy stress rate tensor are given by:

$$\dot{\sigma}_{ij} = C_{ijkl}\dot{\varepsilon}_{kl} + \sigma_{ip}\dot{\omega}_{pj} + \sigma_{jp}\dot{\omega}_{pi} + \sigma_{ik}\dot{\varepsilon}_{jk} + \sigma_{jk}\dot{\varepsilon}_{ik} - \sigma_{ij}\dot{\varepsilon}_{kk} \quad (i,j,k,l,p = 1,2,3) \quad (34)$$

where C_{ijkl} is a fourth order tensor containing are the elastic components of the constitutive matrix and

$$\dot{\sigma}_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right) \quad and \quad \dot{\omega}_{ij} = \frac{1}{2}\left(v_{i,j} - v_{j,i}\right) \quad (35)$$

In (34) and (35) the superposed dot indicates derivative with respect to time. Expression (34) may be also written as (Hughes and Winget, 1980):

$$\dot{\sigma}_{ij} = \left(C_{ijkl} + \hat{C}_{ijkl}\right)\dot{\varepsilon}_{kl} + W_{ijkl}\dot{\omega}_{kl} \quad (i,j,k,l = 1,2,3) \quad (36)$$

with

$$\hat{C}_{ijkl} = -\sigma_{ij}\delta_{kl} + \frac{1}{2}\left(\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} + \sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il}\right) \quad (37)$$

and

$$W_{ijkl} = \frac{1}{2}\left(\sigma_{il}\delta_{jk} + \sigma_{jl}\delta_{ik} - \sigma_{ik}\delta_{jl} - \sigma_{jk}\delta_{il}\right) \quad (38)$$

In matrix form the constitutive equations are given by:

$$\dot{\sigma} = \left(C + \hat{C}\right) + W \dot{\varepsilon} = \left[\left(C + \hat{C}\right) \ W\right]\begin{Bmatrix}\dot{\varepsilon} \\ \dot{\omega}\end{Bmatrix} \quad (39)$$

where

$$\hat{C} = \begin{vmatrix}
\sigma_{11} & -\sigma_{11} & -\sigma_{11} & \vdots & \sigma_{12} & 0 & \sigma_{13} \\
-\sigma_{22} & \sigma_{22} & -\sigma_{22} & \vdots & \sigma_{12} & \sigma_{23} & 0 \\
-\sigma_{33} & -\sigma_{33} & \sigma_{33} & \vdots & 0 & \sigma_{23} & \sigma_{13} \\
\cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\
0 & 0 & -\sigma_{12} & \vdots & \frac{(\sigma_{11}+\sigma_{22})}{2} & \frac{\sigma_{13}}{2} & \frac{\sigma_{23}}{2} \\
-\sigma_{23} & 0 & 0 & \vdots & \frac{\sigma_{13}}{2} & \frac{(\sigma_{22}+\sigma_{33})}{2} & \frac{\sigma_{12}}{2} \\
0 & -\sigma_{13} & 0 & \vdots & \frac{\sigma_{23}}{2} & \frac{\sigma_{12}}{2} & \frac{(\sigma_{33}+\sigma_{11})}{2}
\end{vmatrix}$$

$$and \ W = \begin{bmatrix}
\sigma_{12} & 0 & -\sigma_{13} \\
-\sigma_{12} & \sigma_{23} & 0 \\
0 & -\sigma_{23} & -\sigma_{13} \\
\frac{(\sigma_{22}-\sigma_{11})}{2} & \frac{\sigma_{13}}{2} & -\frac{\sigma_{23}}{2} \\
-\frac{\sigma_{13}}{2} & \frac{(\sigma_{33}-\sigma_{22})}{2} & \frac{\sigma_{12}}{2} \\
\frac{\sigma_{23}}{2} & -\frac{\sigma_{12}}{2} & \frac{(\sigma_{11}-\sigma_{33})}{2}
\end{bmatrix}$$

(40)

In expression (39) $\hat{C}$ and W, given by (40), correspond to the following order of strain rate and rotation rate components:

$$\left\{ \dot{\varepsilon}^T, \quad ^r \right\} = \left\{ \dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, \dot{\varepsilon}_{33}, \dot{\varepsilon}_{12}, \dot{\varepsilon}_{23}, \dot{\varepsilon}_{31}, \dot{\omega}_{12}, \dot{\omega}_{23}, \dot{\omega}_{31} \right\} \quad (41)$$

If the term $\sigma_{ij}\,\dot{\sigma}_{kk}$ is neglected in (34), the matrix C becomes a symmetric matrix given by:

$$\hat{C} = \begin{bmatrix} 2\sigma_{11} & 0 & 0 & \sigma_{12} & 0 & \sigma_{13} \\ & 2\sigma_{22} & 0 & \sigma_{12} & \sigma_{23} & 0 \\ & & 2\sigma_{33} & 0 & \sigma_{23} & \sigma_{13} \\ \hline & & & \dfrac{(\sigma_{11}+\sigma_{22})}{2} & \dfrac{\sigma_{13}}{2} & \dfrac{\sigma_{23}}{2} \\ & Symmetric & & & \dfrac{(\sigma_{22}+\sigma_{33})}{2} & \dfrac{\sigma_{12}}{2} \\ & & & & & \dfrac{(\sigma_{33}+\sigma_{11})}{2} \end{bmatrix} = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \quad (42)$$

The constitutive equation, which will be used to obtain the internal force, may be written in an incremental form as:

$$\begin{Bmatrix} \Delta\bar{\sigma} \\ 0 \end{Bmatrix}_{9x1} = \begin{bmatrix} (C + \hat{C})_{6x6} & W_{6x3} \\ 0_{3x6} & 0_{3x3} \end{bmatrix} \begin{Bmatrix} \Delta\bar{\varepsilon} \\ \Delta\bar{\omega} \end{Bmatrix} = A(\sigma)\Delta\hat{\varepsilon} \quad (43)$$

where $\Delta\hat{\varepsilon}^T = \left\{ \Delta\bar{\varepsilon}^T, \Delta\bar{\omega}^T \right\}$.

Expression (43) results in a non symmetric matrix, which is an important drawback in the evaluation of the stiffness matrix for static or dynamic analysis. Therefore, it is convenient to work with the following constitutive equations:

$$\begin{Bmatrix} \Delta\bar{\sigma} \\ \Delta\bar{\sigma}^* \end{Bmatrix} = \begin{bmatrix} (C + \hat{C})_{6x6} & W_{6x3} \\ W^T_{3x6} & \overline{C}_{3x3} \end{bmatrix} \begin{Bmatrix} \Delta\bar{\varepsilon} \\ \Delta\bar{\omega} \end{Bmatrix} = T(\sigma)\Delta\hat{\varepsilon} \quad (44)$$

where $T(\sigma)$ is a symmetric matrix (also used by Liu, 1981). In expression (45) $\overline{C}$ has the same form as C_{22}, but with negative values in the off-diagonal terms.

The internal force increment is given by:

$$\Delta F_{int} = \int_{\Omega} \tilde{B}^T \Delta\bar{\sigma}\, d\Omega = \int_{\Omega} \tilde{B}^T T(\sigma)\Delta\hat{\varepsilon}\, d\Omega \quad (45)$$

where

$$B = \begin{bmatrix} \bar{B}(0) \\ \text{given by (26)} \\ \hline \bar{b}_2^T & -\bar{b}_1^T & 0 \\ 0 & \bar{b}_3^T & -\bar{b}_2^T \\ -\bar{b}_3^T & 0 & \bar{b}_1^T \end{bmatrix} + \begin{vmatrix} \bar{B}^{HC}(\xi,\eta,\varsigma) \\ \text{given by (26)} \\ \hline \frac{1}{8}G_{22}\varsigma\gamma_3^T & -\frac{1}{8}G_{11}\varsigma\gamma_3^T & 0 \\ 0 & \frac{1}{8}G_{33}\xi\gamma_2^T - \frac{1}{8}G_{22}\xi\gamma_1^T \\ -\frac{1}{8}G_{33}\eta\gamma_3^T & 0 & \frac{1}{8}G_{11}\eta\gamma_1^T \end{vmatrix} \tag{46}$$

and

$$T(\sigma) = \begin{bmatrix} C_{6x6} & 0_{6x3} \\ 0_{3x6} & 0_{3x3} \end{bmatrix} + \begin{bmatrix} 2\sigma_{11} & 0 & 0 & \sigma_{12} & 0 & \sigma_{13} & \sigma_{12} & 0 & -\sigma_{13} \\ & 2\sigma_{22} & 0 & \sigma_{12} & \sigma_{23} & 0 & -\sigma_{12} & \sigma_{23} & 0 \\ & & 2\sigma_{23} & 0 & \sigma_{23} & \sigma_{13} & 0 & -\sigma_{13} & -\sigma_{13} \\ & & & \frac{(\sigma_{11}+\sigma_{22})}{2} & \frac{\sigma_{13}}{2} & & \frac{\sigma_{23}}{2} & \frac{(\sigma_{22}-\sigma_{11})}{2} & \frac{\sigma_{13}}{2} & -\frac{\sigma_{23}}{2} \\ & & & & \frac{(\sigma_{22}+\sigma_{33})}{2} & \frac{\sigma_{12}}{2} & -\frac{\sigma_{13}}{2} & \frac{(\sigma_{33}-\sigma_{22})}{2} & \frac{\sigma_{12}}{2} \\ & & & & & \frac{(\sigma_{33}+\sigma_{11})}{2} & \frac{\sigma_{23}}{2} & -\frac{\sigma_{12}}{2} & \frac{(\sigma_{11}-\sigma_{33})}{2} \\ & \text{Symmetric} & & & & & \frac{(\sigma_{11}+\sigma_{22})}{2} & -\frac{\sigma_{13}}{2} & -\frac{\sigma_{23}}{2} \\ & & & & & & & \frac{(\sigma_{22}+\sigma_{33})}{2} & -\frac{\sigma_{12}}{2} \\ & & & & & & & & \frac{(\sigma_{33}+\sigma_{11})}{2} \end{bmatrix} \tag{47}$$

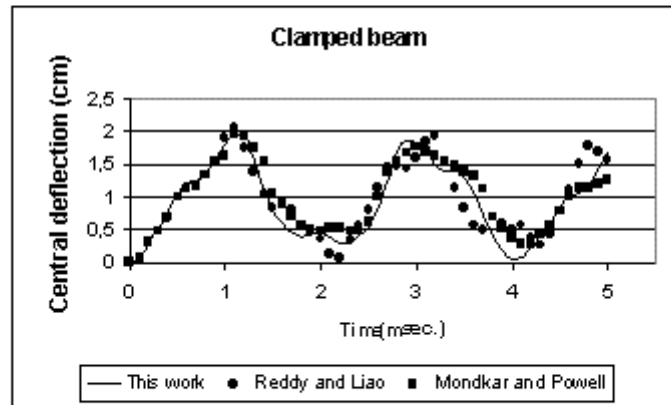In Eq. (47) the last matrix is the initial-stress matrix.

Numerical Results

Example 1 – Nonlinear transient response of a clamped beam

The geometry and the finite element mesh of the beam are shown in Fig.3a. The material properties used in the example are: E= 2.0683×10^{11} N/m^2 , $\rho = 2.714 \times 10^3$ kg/m^3 and $\nu = 0$.

a)

b)

Fig. 3 a) Geometry of the beam   b) Transient response comparison

The intensity of the step load is P(t)= 2846.72 N and the time step used is Δt= 0.09 μ. The transient response is presented in Fig. 3b and is compared with the results obtained by Liao and Reddy (1987) using three-node beam elements and by Mondkar and Powell (1977) employing eight-node plane stress elements. In both References a Total Lagrangian formulation with the second Piola-Kirchoff stress tensor were used.

Example 2 – Nonlinear dynamic analysis of a simply-supported stiffened plate subjected to uniform pressure

The geometry of the stiffened plate is shown in Fig. 4. The material properties are: E= 2.0683×10^{11} N/m^2 , $\rho = 2.714 \times 10^3$ kg/m^3 and $\nu = 0.3$. The intensity of the uniform step load is 25850 N/m^2 and a time interval Δt 0.15 μ sec was adopted. Results are compared with those obtained by Liao and Reddy (1987) using nine-node shell elements for the plate and three-node beam elements for the stiffeners.
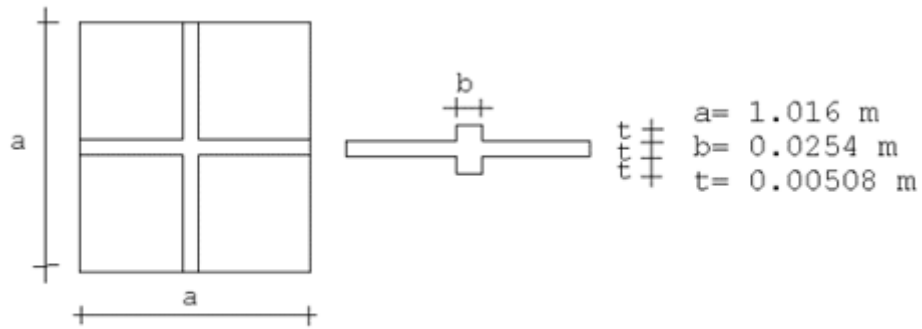
Fig. 4 Geometry of the stiffened plate

a= 1.016 m
b= 0.0254 m
t= 0.00508 m

In order to study the influence of mesh distortion, one quarter of the non-stiffened plate was modeled with an undistorted and a non uniform mesh, which are shown in Figs. 5a and 5b, respectively. Results obtained with these meshes are shown in Fig. 6. Finally, the transient response for the same plate, but with stiffeners, is presented in Fig. 7 and compared with the results obtained by Liao and Reddy (1987).
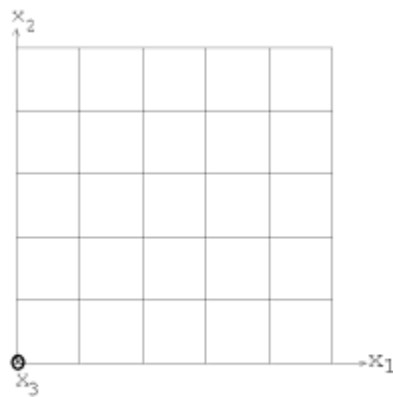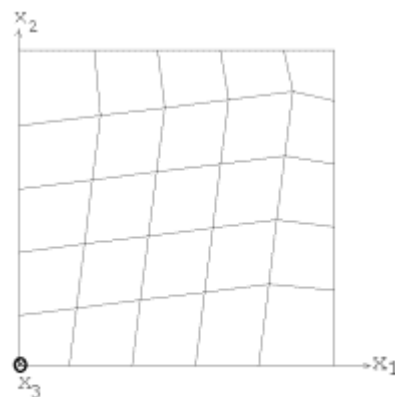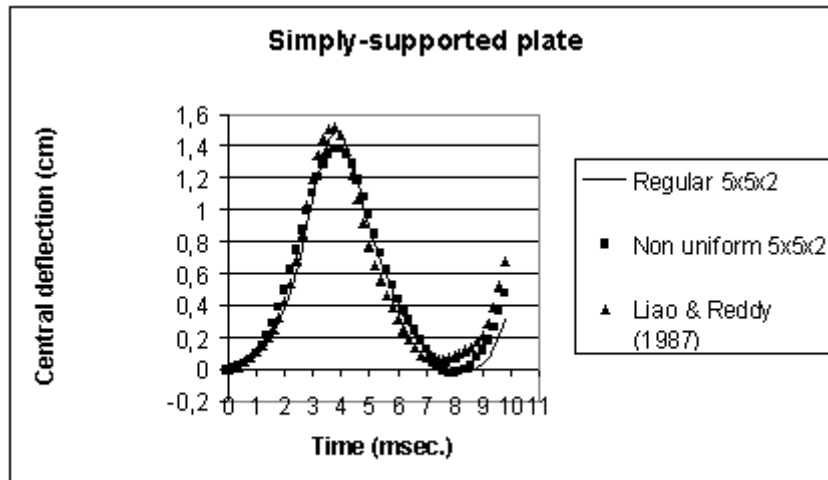

Fig.   5a) Uniform mesh

Fig. 5b) Non uniform mesh

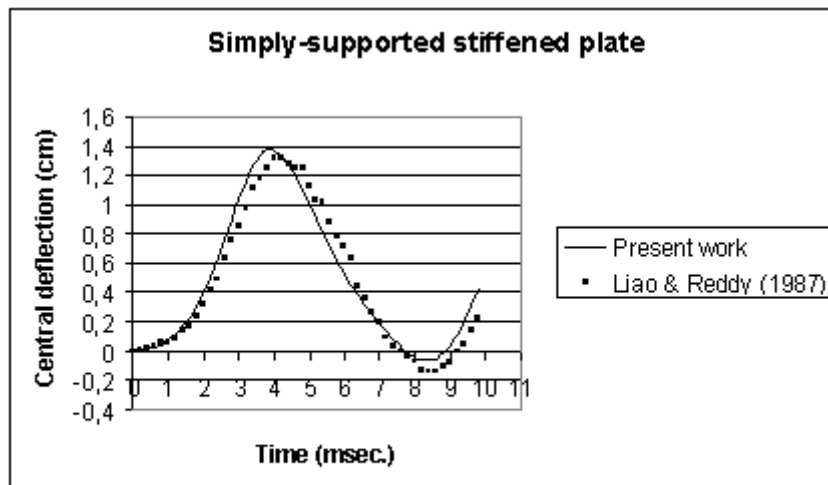**Fig. 6   Responses for a regular and non uniform mesh**

Fig. 7   Transient response for simply-supported stiffened case

Example 3 – Nonlinear dynamic analysis of a clamped spherical shell under a concentrated apex load

The geometry, material properties and finite element mesh are shown in Fig. 8. The nonlinear transient response and a comparison with the results obtained by Bathe et ali. (1974), using eight-node axisymmetric elements and an implicit scheme and a Total Lagrangian formulation with the Second Piola-Kirchoff stress tensor, is shown in Fig. 9.

$E = 6.894 \times 10^{10} \, N/m^2$
$P = 111.2 \, N$
$\rho = 2618 \, kg/m^3$
$\nu = 0.3$
$\Delta t = 0.0027 \, \mu sec.$
$R = 0.1209 \, m$
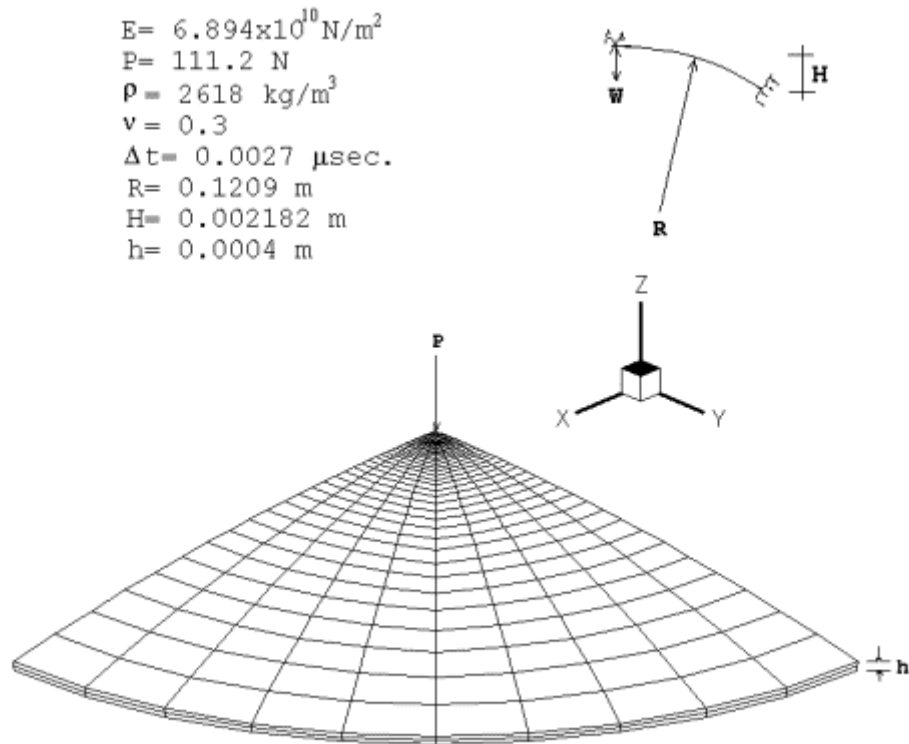$H = 0.002182 \, m$
$h = 0.0004 \, m$

Fig. 8 Geometry, properties and finite element mesh of the clamped spherical shell (one quarter)
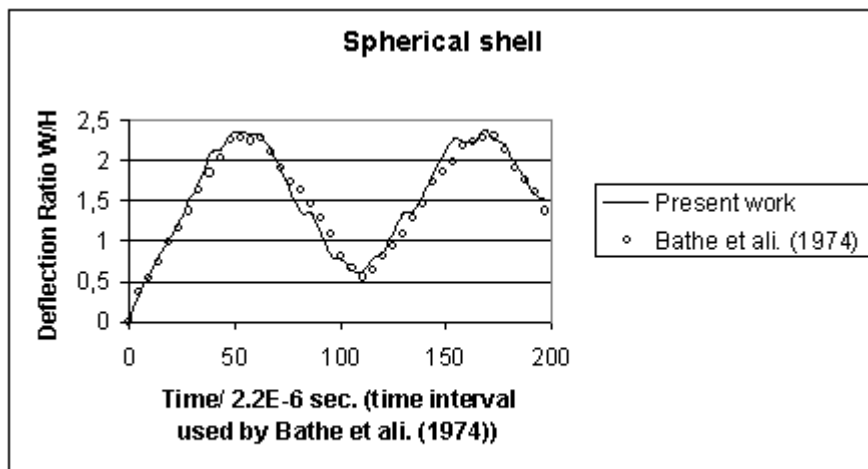

Fig. 9   Nonlinear dynamic response of a spherical shell

Conclusions

Good results were obtained in terms of accuracy and computational performance (about 800 Mflops in a Cray T-94 supercomputer) for all the examples included in this work.

In comparison with explicit time integration schemes conventionally employed in computational structural dynamics, the Taylor-Galerkin scheme used here is very fast

and suitable for vectorization . It is not necessary to evaluate $\mathbf{u}^{-\Delta\tau}$ in t=0 to start the time marching process, as required in the central difference method (Bathe,1996).

The underintegrated element employed in this work is very efficient to solve 3-D problems and it behaves well when applied to the analysis of plates and shells, removing shear and membrane locking, which are usual drawbacks when low-order elements with reduced integration are used. Furthermore, the stabilization matrix does not depend from any parameter defined by the user as works based in Flanagan and Belytschko (1981) and on any material property as in the formulation used by Belytschko and Binderman (1993).

The rate-type constitutive equation behaves well and material nonlinearity (plasticity or viscoplasticity) may be easily included. This aspect is favored by the fact that four integration points have been used, allowing a better representation of plastification fronts than those obtained with a one-point quadrature.

More studies in finite displacement analysis, inclusion of material nonlinearities, the analysis of plates and shells of composite materials and the implementation of implicit schemes are left for future work.

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Nomenclature

$\overline{b}_i$ = volumetric average of the shape function gradient in x_i direction

$\widetilde{B}$ = matrix relating strain and spin tensor components to displacements in a co-rotational system

C = constitutive matrix containing material properties

f_i = body force components per unit volume

F_{int} = internal force vector

G = inverse of the Jacobian matrix

h_i = vectors identifying spurious mode patterns

$J(0)$ = Jacobian matrix evaluated at the element center

$|J(0)|$ = determinant of the Jacobian matrix

M_C = consistent mass matrix

M_L = lumped mass matrix

N = vector containing the shape functions

$\overline{p}_i$ = surface loads components

P_f = load vector due to body forces

P_p = load vector due to surface forces

R = rotation matrix to transform vector and matrices from the global system of reference to the co-rotational system

$\mathbf{t}$ = time

T = constitutive matrix

u_i = displacement components

v_i = velocity components

x_1, x_2, x_3 = space global coordinates

$\hat{x}_1, \hat{x}_2, \hat{x}_3$ = space co-rotational coordinates

Δt = time interval

ε_{ij} = strain tensor components

γ_i = stabilization vectors

ξ, η, ζ = curvilinear coordinates

μ = damping coefficient

ω_{ij} = spin tensor components

Ω = problem domain

Ω_e = finite element volume

ρ = specific mass

σ_{ij} = stress tensor components

σ_{ij}^{dev} = deviatoric part of the stress components

Γ_v = part of the surface boundary where velocities are prescribed

Γ_σ = part of the surface boundary where loads are prescribed