Wilson Ratio of Fermi Gases in One Dimension

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We calculate the Wilson ratio of the one-dimensional Fermi gas with spin imbalance. The Wilson ratio of attractively interacting fermions is solely determined by the density stiffness and sound velocity of pairs and of excess fermions for the two-component Tomonaga-Luttinger liquid phase. The ratio exhibits anomalous enhancement at the two critical points due to the sudden change in the density of states. Despite a breakdown of the quasiparticle description in one dimension, two important features of the Fermi liquid are retained; namely, the specific heat is linearly proportional to temperature, whereas the susceptibility is independent of temperature. In contrast to the phenomenological Tomonaga-Luttinger liquid parameter, the Wilson ratio provides a powerful parameter for testing universal quantum liquids of interacting fermions in one, two, and three dimensions.

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Fermi liquid theory describes the low-energy physics of interacting fermions, conduction electrons, heavy fermion metals, and liquid ^{3}He [1]. It is remarkable that the Wilson ratio, defined as the ratio of the magnetic susceptibility χ to specific heat c_v divided by temperature T

$$R_W = \frac{4}{3}\left(\frac{\pi k_B}{\mu_B g}\right)^2 \frac{\chi}{c_v/T}, \tag{1}$$

is a constant at the renormalization fixed point of these systems. Here, k_B is the Boltzmann constant, μ_B is the Bohr magneton, and g is the Landé factor. For example, $R_W = 1$ for noninteracting or weakly correlated electrons in metals [1], and $R_W = 2$ in the Kondo regime for the impurity problem [2]. The dimensionless Wilson ratio quantifies the interaction effect and spin fluctuations and thus presents a characteristic of strongly correlated Fermi liquids [1]. $R_W > 1$ in strongly correlated systems where the spin fluctuations are enhanced while charge fluctuations are suppressed.

The Wilson ratio has recently been measured in experiments on a gapped spin-1/2 Heisenberg ladder [3]. This opens up the opportunity to probe and understand the universal nature of one-dimensional (1D) quantum liquids through the measurable Wilson ratio. Early calculations of R_W for 1D correlated electrons were considered only in the scenario of spin-charge separation [4,5]. As far as the low-energy physics is concerned, the fixed point critical Tomonaga-Luttinger liquid (TLL) behaves much like the Fermi liquid [6]. For instance, the Wilson ratio of the quasi-1D spin-1/2 Heisenberg ladder near the critical point indicates a single-component TLL with $R_W = 4$ K, where K is the TLL parameter. Moreover, the Wilson ratio is always less than 2 as the band fillings tend towards the Mott insulator in the 1D repulsive Hubbard model [5]. For the 1D spin-1/2 Heisenberg chain, $R_W = 2$ as $T \to 0$ [7]. Here, the Fermi liquid nature arises because the elementary excitations at low temperatures are spinons which are regarded as fermions.

Motivated by the experimental results for the spin ladder [3], we consider the Wilson ratio in the context of the spin-1/2 delta-function interacting Fermi gas [8,9]. The quantum liquids exhibited by this model include the paradigm of a spin-charge separated TLL in the repulsive regime and a two-component TLL of pairs and single fermions in the attractive regime. The pairing phase has attracted a great deal of attention [10–16], with the key features of the $T = 0$ pairing phase [17–19] experimentally confirmed using finite temperature density profiles of trapped fermionic ^{6}Li atoms [20,21].

In this context, the Wilson ratio of the 1D attractive Fermi gas with polarization is particularly interesting due to the coexistence of pairing and depairing under the external magnetic field. It is natural to ask if the Wilson ratio can capture a similar Fermi liquid nature of such a particular pairing phase. Here, we report our key result for the attractive Fermi gas

$$R_W = \frac{4}{(v_N^b + 4v_N^u)(\frac{1}{v_s^b} + \frac{1}{v_s^u})}, \quad (2)$$

which holds throughout the two-component TLL phase. This result is in terms of the density stiffness $v_N^{b,u}$ and sound velocity $v_s^{b,u}$ for pairs b and excess single fermions u. These parameters can be calculated from the ground state energy. Figure 1 shows that at finite temperatures, the contour plot of R_W can map out not only the two-component TLL phase but also the quantum criticality of the attractive Fermi gas. The Wilson ratio thus gives a simple testable parameter to quantify interaction effects and the competing order between pairing and depairing.

The model.—The δ-interacting spin-1/2 Fermi gas with $N = N_\uparrow + N_\downarrow$ fermions of mass m with external magnetic field H is described by the Hamiltonian [8,9,21]

$$\mathcal{H} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g_{1D}\sum_{i=1}^{N_\uparrow}\sum_{j=1}^{N_\downarrow}\delta(x_i - x_j) + E_z \quad (3)$$

in which the terms are the kinetic energy, interaction energy, and Zeeman energy $E_z = -(1/2)g\mu_B H(N_\uparrow - N_\downarrow)$. Here, the intercomponent interaction is determined by an effective 1D scattering length $g_{1D} = -(2\hbar^2/ma_{1D})$ which can be tuned from the weakly interacting regime ($g_{1D} \to 0^{\pm}$) to the strong coupling regime ($g_{1D} \to \pm\infty$) via Feshbach resonances and confinement-induced resonances [22]. $g_{1D} > 0$ (< 0) is the contact repulsive (attractive) interaction. The total density $n = n_\uparrow + n_\downarrow$, the magnetization $M = (n_\uparrow - n_\downarrow)/2$, and the polarization $P = (n_\uparrow - n_\downarrow)/n$, where $n = N/L$ is the linear density and L is the length of the system. For convenience, we define the interaction strength as $c = mg_{1D}/\hbar^2$ and dimensionless parameter $\gamma = c/n$ for physical analysis. We set Boltzmann constant $k_B = 1$ and $\mu_B g = 1$.

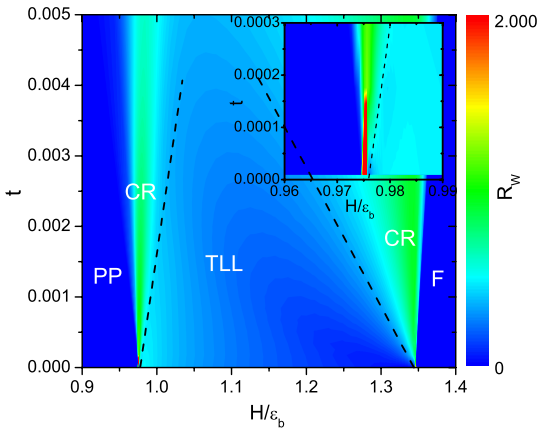

FIG. 1 (color online). Contour plot of the Wilson ratio R_W [Eq. (1)] of the attractive Fermi gas for dimensionless interaction $|\gamma| = 10$ as a function of the reduced temperature $t = T/\varepsilon_b$ and magnetic field. ε_b is the binding energy. The result [Eq. (2)] provides a criterion for the two-component TLL phase in the region below the dashed lines, where R_W is temperature independent. The dashed lines indicate the crossover temperature $T^* \sim |H - H_c|$ separating the relativistic liquid from the nonrelativistic liquid. $R_W = 0$ for both the TLL of pairs (PP) and the TLL of excess fermions (F). In the critical regimes (CR), R_W gives a temperature-dependent scaling. However, near the two critical points, the ratio reveals anomalous enhancement, discussed further in the text. The inset shows the enhancement at the lower critical point.

The thermodynamic properties of the model are determined by the thermodynamic Bethe ansatz (TBA) equations [23]. A high precision equation of state in the physically interesting low temperature and strong coupling regime ($T \ll \epsilon_b$, H and $\gamma \gg 1$) has been derived [24,25]. The hydrodynamic description of the attractive gas [Eq. (3)] is restricted to the limit cases $c \to -\infty$ and $c \to 0^-$ [26].

Susceptibility.—In the Fermi liquid, the interaction enters the susceptibility and specific heat via the effective mass and the Landau parameters [27]. Thus, the specific heat increases linearly with the temperature T because only the electrons within $k_B T$ near the Fermi surface contribute to the specific heat. The susceptibility is independent of temperature since only the electrons within $\mu_B gH$ near the Fermi surface contribute to the magnetization. This is a consequence of the forward scattering process between quasiparticles near the Fermi surface. In contrast, in 1D many-body systems, all particles participate in the low-energy physics and thus form collective motion of bosons, i.e., the TLL. However, the TLL is also the consequence of the forward scattering process involving low-lying excitations close to Fermi points. Therefore, it is natural to expect that 1D many-body systems have a Fermi liquid nature in the low-energy sector.

Here, we find such a Fermi liquid signature of the 1D Fermi gas using the analytic results for the susceptibility and specific heat obtained via the TBA equations [28]. At zero temperature, the susceptibility can be calculated from the dressed energy equations which are obtained from the TBA equations in the limit $T \to 0$ [28]. The dressed energy equations give the full phase diagram and magnetic properties in the grand canonical ensemble.

For values of the magnetic field between the lower and upper critical fields H_{c1} and H_{c2}, the zero temperature susceptibility of the gapless phase can be expressed in the form

$$\frac{1}{\chi} = \frac{1}{\chi_u} + \frac{1}{\chi_b}. \quad (4)$$

This result can be established on general grounds. The effective magnetic field H depends on the chemical potential bias $H := \Delta\mu = \mu_\uparrow - \mu_\downarrow$. The magnetization depends on the difference $\Delta n = n_\uparrow - n_\downarrow$. We prove that the magnetic susceptibility $\chi = (1/2)\partial\Delta n/\partial\Delta\mu$ can be written in terms of the charge susceptibilities of bound pairs and excess fermions $\chi_{b,u} = (1/2)\partial n_{b,u}/\partial\mu_{b,u}|_{\mu_{u,b}}$, where

$\mu_b = \mu + \epsilon_b/2$, $\mu_u = \mu + H/2$, and the total density n is fixed. Here, n_b and n_u are the densities of pairs and excess fermions. Physically, the system has two processes occurring in parallel, namely, the breaking of pairs and the alignment of spins. The analog for the zero temperature susceptibility of the gapless phase is thus two parallel resistors in a circuit.

We also find that the effective susceptibilities for the TLL of bound pairs and the TLL of excess fermions are expressed as $\chi_b = 1/(\hbar\pi v_N^b)$ and $\chi_u = 1/(4\hbar\pi v_N^u)$. The density stiffness parameters are obtained from $v_N^r = (L/\pi\hbar)(\partial^2 E_0^r/\partial N_r^2)$ for a Galilean invariant system, with $r = 1$ for excess fermions and $r = 2$ for bound pairs. For the strongly interacting regime ($\gamma > 1$), the ground state energies for the pairs and excess fermions are given explicitly by [19] $E_0^r \approx (\hbar^2/2m)(\pi^2 N^3/3rL^2)(1 + (2A_r/|c|) + (3A_r^2/c^2))$ with $A_1 = 4n_2$ and $A_2 = 2n_1 + n_2$. Here, n_1 and n_2 are the density of excess fermions and pairs, respectively. Thus,

$$v_N^b = \frac{\hbar\pi n_2}{2m}\left[1 + \frac{4}{|c|}(n - 3n_2) + \frac{3}{c^2}(4n^2 - 24nn_2 + 30n_2^2)\right],$$
$$v_N^u = \frac{\hbar\pi n_1}{m}\left[1 + \frac{4}{|c|}(n - 2n_1) + \frac{4}{c^2}(3n^2 + 10n_1^2 - 12nn_1)\right].$$

The analytic expression (4) with these velocities is in excellent agreement with the numerical results (see the inset in Fig. 2).

The onset susceptibility at the lower and upper critical fields H_{c1} and H_{c2} is related to the collective nature of the pairs and excess fermions, with

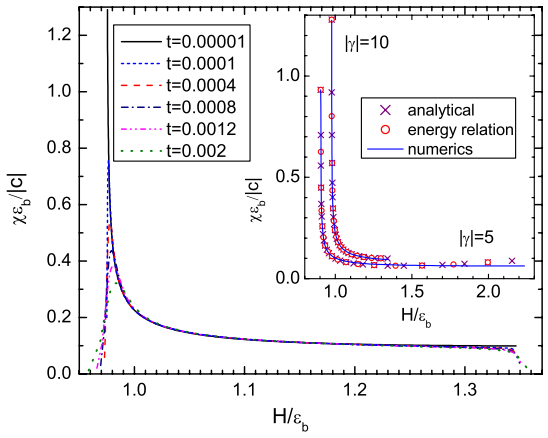

FIG. 2 (color online). The dimensionless susceptibility vs magnetic field for $|\gamma| = 10$ at different temperatures. The susceptibility is independent of temperature for $T < H - H_{c1}$ and $T < H_{c2} - H$. Round peaks of the susceptibility in the vicinity of the two critical points are observed at low temperatures. The inset shows the susceptibility for $|\gamma| = 5$ and 10 at $T = 0$. The pink crosses denote the analytic result [Eq. (4)] which is in excellent agreement with the numerical results obtained from the field-magnetization relation [19] (red circles) and from the dressed energy equations [28] (blue lines).

$$\chi|_{H\to H_{c1}+0} = \frac{1}{\hbar\pi v_N^b}\Big|_{n_2=n/2} = \frac{K^b}{\hbar\pi v_s^b}\Big|_{n_2=n/2}, \quad (5)$$

$$\chi|_{H\to H_{c2}-0} = \frac{1}{4\hbar\pi v_N^u}\Big|_{n_1=n} = \frac{K^u}{4\hbar\pi v_s^u}\Big|_{n_1=n}. \quad (6)$$

Here, v_s^r and $K^r = v_s^r/v_N^r$ are the sound velocities and effective TLL parameters of the bound pairs and excess single fermions. From the relation $v_s^r = \sqrt{(L/mnr)(\partial^2 E_0^r/\partial L^2)}$, the velocities are given by $v_s^r = (\hbar/2m)(2\pi n_r/r)(1 + 2A_r/|c| + 3A_r^2/c^2)$.

The separation of the susceptibility [Eq. (4)] naturally suggests that the low-energy physics of the polarized pairing phase is described by a renormalization fixed point of the two-component TLL class, where the interaction effect enters into the collective velocities, or equivalently the effective masses of the two TLLs are varied by the interaction. At finite low temperatures, the two-component TLL acquires a universal form $F(T, H) \approx E_0(H) - (\pi k_B^2 T^2/6\hbar)(1/v_s^b + 1/v_s^u)$ of the free energy. For temperature $T < H - H_{c1}$ and $T < H_{c2} - H$, the susceptibility is indeed independent of temperature, provided that $-\partial^2(1/v_s^b + 1/v_s^u)/\partial H^2 \approx 0$; see Fig. 2. We clearly see that the $T = 0$ divergent susceptibility near the critical point H_{c1} evolves into round peaks at low temperatures. The peak hight decreases as the temperature increases. Here, the leading irrelevant operators gives a correction of the order $O(T^2)$ to the low energy in the vicinities of the two critical points.

For the quantum critical regime ($T > H - H_{c1}$ and $T > H_{c2} - H$), the susceptibility defines the universality class for quantum criticality of nonrelativistic Fermi theory, with [28]

$$\chi \sim \frac{|c|}{\epsilon_b}[\lambda_0 + \lambda_s t^{d/z+1-2/\nu z}\mathrm{Li}_{-1/2}(-e^{[\alpha(h-h_{c1})]/t^{1/\nu z}})]. \quad (7)$$

Near the critical point $h_{c1} = -2\tilde{\mu} + (32/3\pi\sqrt{2}) \times (\tilde{\mu} + 1/2)^{3/2}$, we have $\lambda_0 = 0$ and $\lambda \approx 1/(8\sqrt{2}\pi) \times [1 - (6/\pi)\sqrt{(h - h_{c1})/2}]$ with $\alpha = 1/2$, $t = T/\epsilon_b$, and $h = H/\epsilon_b$. Here, the dynamical critical exponent $z = 2$ and correlation length exponent $\nu = 1/2$ for different phases of the spin states. Near the upper critical point h_{c2}, the susceptibility defines a similar form as Eq. (7), but with the background susceptibility $\lambda_0 \neq 0$ [28].

Specific heat.—We turn now to the specific heat of the attractive Fermi gas. The low temperature expansion of the TBA equations with respect to $T \ll H$, ϵ_b gives

$$c_v = \frac{\pi k_B^2 T}{3\hbar}\left(\frac{1}{v_s^b} + \frac{1}{v_s^u}\right). \quad (8)$$

The linear T dependence of the specific heat is a consequence of linear dispersions in branches of pairs and single fermions. The breakdown of this linear temperature-dependent relation defines a crossover

temperature T^* which characterizes a universal crossover from a relativistic dispersion into a nonrelativistic dispersion [24,29].

We see clearly in Fig. 3 that at low temperatures, a peak evolves in the specific heat near each of the two critical points, i.e., near $P = 0$ and $P = 1$ due to a sudden change in the density of states. We also note that the peak positions mark the TLL specific heat curve [Eq. (8)]. The two peaks merge at the top of the TLL phase in Fig. 1. Thus, the peak position in turn gives the TLL phase boundary in the $c_v - P$ or $c_v - H$ plane. The specific heat obtained from the equation of state [25] also defines a scaling behavior

$$c_v \sim \sqrt{\frac{2m\varepsilon_b}{\hbar^2 t^2}}[\nu_0 + \nu_s t^{d/z+1-2/\nu z}\mathrm{Li}_{-1/2}(-e^{[\alpha(h-h_c)]/t^{1/\nu z}})], \tag{9}$$

where ν_0, ν_s, and α are constants which can be determined from the closed form of the specific heat if necessary [28]. The two-component TLL specific heat [Eq. (8)] is clearly manifest in Fig. 3 from the numerical result obtained using the equation of state.

Wilson ratio.—The linear temperature-dependent nature of the specific heat and the separable feature of the susceptibility give the Wilson ratio [Eq. (2)] for the effective low-energy physics of the two-component TLL. This Wilson ratio for the 1D attractive Fermi gas is significantly different from the ratio obtained for the field-induced gapless phase in the quasi-1D gapped spin ladder [3], where the gapless phase is a single-component TLL [4,6] and the ratio gives a renormalization fixed point of a linear spin-1/2 chain in zero field. It is interesting to note that for the 1D attractive Fermi gas, the onset Wilson ratio also depends solely on the TLL parameters, with

$$W_R|_{H\to H_{c1}} = 4K^b|_{n_2\to n/2}, \qquad W_R|_{H\to H_{c2}} = K^u|_{n_1\to n}.$$

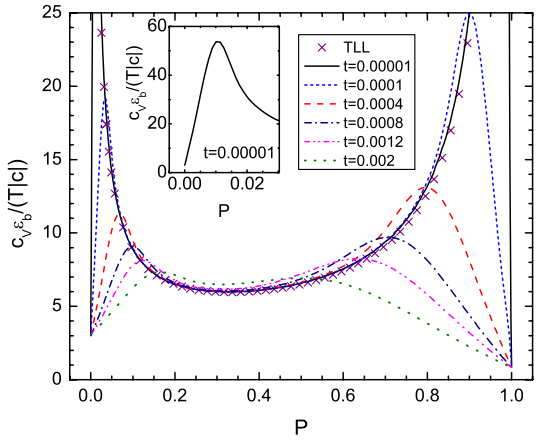

FIG. 3 (color online). Dimensionless specific heat vs polarization for $|\gamma| = 10$ at different temperatures. The deviation from linear temperature dependence [Eq. (8)] (red crosses) indicates the breakdown of the two-component TLL. The inset shows a round peak evolved near H_{c1} at $T = 0.00001\epsilon_b$.

Here, we find

$$K^b \approx 1 + \frac{6}{|c|}n_2 + \frac{3}{c^2}n_2(3n_2 + 4n),$$
$$K^u \approx 1 + \frac{4}{|c|}n_1 + \frac{4}{c^2}n_1(n_1 + 2n). \tag{10}$$

Note that the values in the limit of infinitely strong coupling are $W_R = 4$ at H_{c1} and $W_R = 1$ at H_{c2}.

The anomalous enhancement of the Wilson ratio near the onset values is shown in Fig. 4. Anomalous enhancement of the Wilson ratio has been observed near the metal-insulator transition in simulations of a three-dimensional quantum spin liquid [30]. Here, for the 1D attractive Fermi gases, this anomalous divergence is mainly due to sudden changes in the density of states either in the bound state or excess fermion branch. Again, deviation from the Wilson ratio [Eq. (2)] gives the crossover temperature $T^* \sim |H - H_c|$ separating the TLL from the free fermion liquid near the critical points. In addition to the anomalous divergence of the onset Wilson ratio, a round peak is observed near $P \approx 0.1$ due to the competing ordering of the two TLLs. $R_W < 1$ for finite values of the polarization ($0 < P < 1$).

In contrast to this enhancement, for the repulsive regime, the Wilson ratio is always less than 2, i.e., $R_W = 2/(1 + v_\sigma/v_c)$, which simply gives a fixed point of the TLL in the context of spin-charge separation. Here, the charge and spin velocities $v_{c,\sigma}$ can be calculated following Ref. [31].

The Wilson ratio of 1D Fermi gases can in principle be measured in experiments. The finite temperature density profiles of a 1D trapped Fermi gas of ^{6}Li atoms have been measured [20]. Most recently, the susceptibility has been directly obtained from the density profile of the trapped atomic cloud in higher dimensions [32]. High precision measurements of thermodynamic quantities have also been

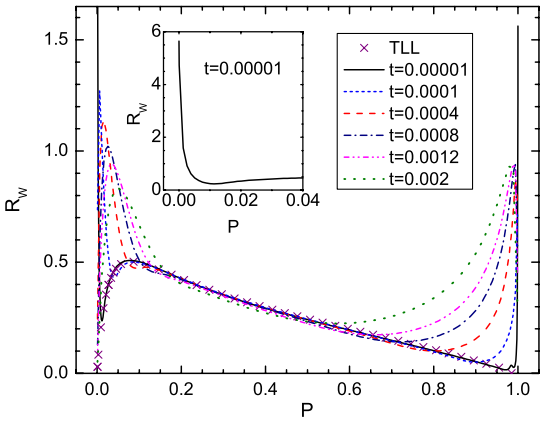

FIG. 4 (color online). Wilson Ration vs polarization for $|\gamma| = 10$ at different temperatures. The numerical result obtained from the equation of state fully agrees with the Wilson ratio [Eq. (2)] (red crosses) for the two-component TLL phase. The deviations from the result [Eq. (2)] characterize the crossover temperature T^*. Anomalous behavior is found near $P = 0$ and $P = 1$ (see the inset for the region near the critical point H_{c1}).

reported [33]. For the 1D case, the predicted susceptibility could be tested from the density profiles $n_{\uparrow,\downarrow}$ and the chemical potential bias.

The Wilson ratio of the 1D attractive Fermi gases which we have obtained provides a measurable parameter to quantify different phases of quantum liquids in 1D interacting fermions with polarization. At low temperatures, the Fermi liquid nature is retained in 1D many-body systems of interacting fermions. Our analysis can be adapted to different systems, such as interacting fermions, bosons, and mixtures composed of cold atoms with higher spin symmetry.

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