The Supersymmetric t-J Model and its Quantum Group Deformation

Inaugural-Dissertation
zur
Erlangung der Doktorwürde
des Fachbereichs Physik
der Freien Universität Berlin

vorgelegt von
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Berlin, April 1993

Doação: da autora

Data do recebimento: 26.03.2008

ABSTRACT

In this work we investigate the supersymmetric t-J model in one dimension. The spectrum of the hamiltonian is obtained by means of the algebraic nested Bethe-ansatz method. Furthermore, we present a detailed analysis of the algebraic structure of the states. By combining the Bethe ansatz with the spl(2,1) supersymmetry of the t-J hamiltonian a complete set of eigenstates is constructed. The ground state and the elementary excitations of the model are investigated. We also introduce a new integrable vertex model, the anisotropic t-J model, which possesses $spl_q(2,1)$ "quantum supergroup" invariance.

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Tag der mündlichen Prüfung : 19.5.1993

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Chapter 1

Introduction and main results

1. INTRODUCTION

The discovery of high-Tc superconductors by Bednorz and Müller in 1986 [1] has given raise to searches for new mechanisms of superconductivity, since this new kind of superconductors (ceramics) are not well described by the BCS theory [2], which is generally accepted for the usual metallic superconductors. The pioneer idea for an alternative model goes back to Anderson [3], who suggested that strongly correlated electron systems near the metal-insulator transition may build up a superconductor state. This comes from the fact that the initial compound originating the new superconductors (e.g. La_2CuO_4) is an insulator which is transformed into a superconductor below a certain critical temperature by partial substitution of La by atoms with a lower valence, like Ba or Sr or by increasing the oxygen content [4]. This process is usually referred as doping the system with holes. There is a great deal of experimental evidence to support the view that the essential feature of this new type of superconductors is the quasi-two dimensional motion of holes in CuO planes [5].

Recently, the t-J model has attracted much interesse in connection with high-Tc superconductivity, since it is one of the simplest models for studying strongly correlated electron systems. This model , proposed by Anderson [6], is obtained from a strong coupling expansion of the Hubbard hamiltonian by excluding two electrons at one site. It describes electrons with nearest-neighbor hopping and spin exchange interaction on a quantum lattice, whose hamiltonian reads

$$\mathcal{H}_{tJ} = P\{-t\sum_{j,\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}P + J\sum_{j}(\vec{S}_j.\vec{S}_{j+1} - \frac{n_j n_{j+1}}{4}), \qquad (1.1)$$

(see next chapter). In principle, the t-J model can be studied for any dimensions and for all ratios of J/t. Although high-Tc cooper-oxide materials are at least two-dimensional systems, it is also instructive to investigate the one-dimensional model. It is relatively simple to explore and, as Anderson recently claimed [7], two-dimensional strongly correlated systems may share some properties of the one-dimensional case. It has been observed that for an appropriate choice of the coupling constants ($J/t = 2$) the one-dimensional t-J model becomes supersymmetric and completely integrable by the Bethe ansatz method.

The concept of integrable systems in physics was originated in the earlier developments of the classical mechanics. According to Liouville, a classical hamiltonian system with N degrees of freedom is called integrable if it possesses a set of N independent integrals of motion commuting with respect to the Poisson brackets. One of the most familiar example of an exactly solvable model in classical mechanics is the two-body Kepler problem, whereas the three body problem is not completely integrable. At quantum level, integrability is given by using commutators instead of Poisson brackets. In general, for integrable quantum models it is possible to evaluate some physical quantities such as the spectrum of commuting quantum integrals of motion without any approximation, even for systems with many degrees of freedom. In fact, this is the case of many lattice models, e.g. the Heisenberg chain.

Although the integrability of quantum systems as discussed here is restricted to one dimension, there are many reasons that turn this study relevant. It serves as a test for computer analysis and analytical methods for realistic systems to which, until now, only numerical calculations (e.g. Monte-Carlo simulations) and perturbative methods may be applied. In addition, a nontrivial solvable model reveals an essence of the phenomena under consideration. For instance, many concepts established in critical phenomena were inspired by the exact solution of the Ising model. From the experimental point of view, we mention that there are some materials which behave like one-dimensional systems [8]. Furthermore, as pointed out by Baxter [9]: "One-dimensional lattice models are relevant and can be solved, so why not do so and see what they tell us ?". Integrable

systems also play an interesting role in mathematics. For example, they provide an explicit realization of algebraic structures, as Lie algebras and quantum groups [10,11].

The concept of quantum groups and quantum algebras emerged as a natural abstraction of certain basic ideas of the quantum inverse scattering method. These new mathematical objects have attracted great interesse due to their relation to non-commutative geometry and the theory of knots and links [12]. In physics, the notion of quantum groups appeared in many fields, such as statistical mechanics and conformal field theory [13]. However, the theory of quantum groups has not shown its full power yet. Deformations of algebraic structures played a very important role in the development of modern physics. In fact, the transition from Galilei relativity to that of Lorentz, as well as the transition from classical to quantum mechanics are nothing more than deformations. Both are associated with the parameters "c" and "$\hbar$". Motivated by these reasons, it can be speculated that these new deformations found mainly in the mathematics may lead to new developments in the physics [14].

One of the most powerful techniques for treating one-dimensional quantum systems or two-dimensional classical statistical models (vertex models) is the Bethe ansatz method. It was first introduced by Bethe in 1931 to solve the isotropic Heisenberg model [15] and was subsequently applied to several other models. The approach was developed further by Baxter, who solved the XYZ-Heisenberg model (or the eight-vertex model) [9,16]. A great impulse in the theory of integrable systems was given by Faddeev, Takhtajan and Sklyanin with the development of the algebraic Bethe ansatz method [17], also called quantum inverse scattering method (QYSM).

This algebraic viewpoint introduced by the russian school provides the solution of additional models which in principle were very difficult to solve using the traditional, or coordinate Bethe ansatz approach. Moreover, it provided an unified framework for treating previous methods of exact solution in different areas of theoretical and mathematical physics : i) method of commuting transfer matrices for two-dimensional lattice models of classical statistical mechanics, as the Ising models and vertex models, investigated by Onsager in 1944 [18], Lieb in 1967 [19] and Baxter in 1972 [16]; ii) method of the factorized S-matrices, introduced by Karowski et al in 1978 [20] and developed later by Zamolodchikov et al. [21].

Basically, through the algebraic Bethe ansatz method the problem of finding the

spectrum of some hamiltonian reduces to the resolution of a system of coupled transcendental equations, the so-called Bethe ansatz equations (BAE). Each eigenvector and eigenvalue is characterized by a set of roots of these equations. In the thermodynamic limit the ground state of the hamiltonian at zero temperature, as well as the elementary excitations correspond to some distribution of roots of the BAE which can be handled as rapidities (momenta) of "quasi-particles". Then, the form of the ground state of the model can be obtained by minimizing the free energy of the system. An important contribution in the analysis of the thermodynamic properties of one-dimensional quantum systems was given by Yang and Yang [22] for the Bose-gas system. They used Bethe ansaztz methods to investigate the system at finite temperature.

There are still open questions related to the Bethe ansatz method. One of the most interesting is the completeness of the Bethe vectors. Since the pioneering work of Bethe [15] and a subsequent work of Faddeev and Takhtajan [23] on the isotropic Heisenberg model, it is known that the Bethe ansatz alone does not provide a complete set of states instead it only determines the highest weight vectors of the underlying SU(2) symmetry group. Recently, Essler, Korepin and Schoutens [24] proved for the one-dimensional Hubbard model that the Bethe states are lowest weight vectors with respect to an SO(4) symmetry. Then, for both models a complete set of vectors can be constructed combining the Bethe ansatz with the SU(2) and SO(4) symmetry groups, respectively. However, for other integrable models there is no proof that a complete family of eigenstates can be obtained by Bethe ansatz methods.

The algebraic Bethe ansatz method is based on a fundamental commutation relation called Yang-Baxter equation

$$S_{12}(v_{12})S_{13}(v_{13})S_{23}(v_{23}) = S_{23}(v_{23})S_{13}(v_{13})S_{12}(v_{12}), \tag{1.2}$$

where v is the spectral parameter (see next chapter). It is a sufficient condition for integrability and leads to a consistent and systematic method to construct solvable models. The Yang-Baxter relation has several physical interpretations. In the theory of the factorized S-matrix, where S is the two-body scattering matrix, the Yang-Baxter equation means that the process of 3-particle scattering is reduced to a sequence of pair collisions which do not depend on the time ordering of the two-body collisions. In this case the Yang-Baxter equation has the name of factorization equation. In the vertex

models of classical statistical physics, the Yang-Baxter equation appears as a condition for the vertex weights S which allows for the exact solution of the corresponding model. In this context it is usually referred as "star-triangle" relation.

The solutions of the Yang-Baxter equation can be classified according to the dependence of the matrix S on the spectral parameter in rational, trigonometric or elliptic. In the rational case, the Yang-Baxter algebra is associated to a simple Lie algebra or Lie group. In the trigonometric case, it is a deformation of a Lie algebra or Lie group called "quantum group" that underlies the Yang-Baxter algebra. The name "quantum group" follows from the fact that these new structures are related to the Lie groups in a similar way as quantum mechanics is related to classical mechanics. The elliptic case is until now not very well understood from the algebraic point of view.

Quantum groups are closely related to Yang-Baxter algebras. Nevertheless, quantum group invariance holds only for integrable vertex models with special choices of boundary conditions. This fact has been observed for the six vertex model with anisotropy (or, equivalently, the spin 1/2 anisotropic Heisenberg model with imaginary boundary conditions) [25,26]. In order to treat more general boundary conditions compared to the usual periodic one, a generalization of the algebraic Bethe ansatz method is required [27,28]. In this construction, in addition to the matrix S defining the vertex weights, two new matrices $K^{\pm}$ that take into account the boundary conditions are introduced. The explicit form of these matrices is determined by the requirement of integrability. Actually, this approach really provides a systematic way to get an integrable model with quantum group invariance. Many other integrable models, as the Toda-chain [28,29], nonlinear Schrödinger equation [28,30], the XY model [31], A_{n-1} vertex models [32] among others, were also considered in connection with special boundary conditions. However, the Bethe ansatz equations were derived only for the XXZ chain [28]. Therefore, the problem of finding the spectrum of open chains has not been solved yet in its full extent and is presently a subject of increasing activity.

2. MAIN RESULTS

In this work we investigate the supersymmetric t-J model in one dimension. We present an explicit construction of the eigenvalues and eigenvectors of the model using the algebraic nested Bethe ansatz method. By this procedure we find a new form of the Bethe ansatz equations and also recover the other two forms, previously obtained by Lai [33], Schlotmann [34], Sutherland [35] and Sarkar [36]. The equations are solved as usual, using the string conjecture.

Furthermore, we give a detailed analysis of the algebraic structure of this model. We prove that the Bethe ansatz states are highest weight vectors of an spl(2,1) superalgebra [37]. Then, by acting with the spl(2,1) shift operators we find additional eigenvectors. After that, the complete family of eigenstates is obtained [38]. The proof is based on counting the number of states which happens to be equal to the dimension of the full Hilbert space, namely 3^L. In addition, we investigate the multiplet structure of the anti-ferromagnetic ground state and some low-lying excitations, as spinons and holons. For example, it turns out that the ground state is a member of a quartet.

The thermodynamic properties of the model are discussed for the system at arbitrary filling in the presence of an external magnetic field. The Fermi levels, the excitation spectrum as well as the dispersion relations of the model are calculated. We also derive the relation between some physical quantities of interest as filling and chemical potential. We mention that an analysis of the excitation spectra was also performed by Bares and Blater [40].

We proceed our work introducing a new integrable vertex model, namely the anisotropic t-J model in one-dimension with quantum supergroup invariance [39], whose hamiltonian reads

$$\begin{aligned}\mathcal{H}^{(q)} = &- P\{\sum_{j=1}^{L-1}\sum_{\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}P \\ &- 2\sum_{j=1}^{L-1}\Big(S^x_jS^x_{j+1} + S^y_jS^y_{j+1} + \cos\gamma\ (S^z_jS^z_{j+1} - \frac{n_jn_{j+1}}{4})\Big) - \cos\gamma\sum_{j=1}^{L} n_j\,. \quad (2.1) \\ &+ i\sin(\gamma)(n_1 - n_L) - i\sin(\gamma)\sum_{j=1}^{L-1}(n_jS^z_{j+1} - S^z_jn_{j+1})\end{aligned}$$

Through a generalization of the construction of refs.[27,28,41] we derive the Bethe ansatz equations of the model. We show that they are related to the BAE obtained using

periodic boundary conditions. Moreover, we propose an alternative approach, where the matrix K^+ defining the boundaries is the Markov trace associated with the supergroup $spl_q(2,1)$. This allows for an easy generalization to other cases. We also show that the model under consideration provides an explicit realization of the quantum supergroup $spl_q(2,1)$ and is in addition $spl_q(2,1)$ invariant.

The thesis is divided as follow. In Chapter 2 we solve the one-dimensional supersymmetric t-J model through the quantum inverse scattering method. Furthermore, a careful analysis of the algebraic structure of the model is presented. The proof of the completeness of the Bethe states is also given. In Chapter 3 we discuss the thermodynamic properties of the t-J hamiltonian. By minimizing the free energy of the system we find the ground state of the hamiltonian. A description of the model at arbitrary filling and in the presence of an external magnetic field is given. The Fermi levels and the excitation spectrum, as well as dispersion relations are calculated. Chapter 4 contains a study of the anisotropic t-J model with quantum supergroup invariance.

Chapter 2

Solution and Algebraic Structure of the Supersymmetric t-J Model

1. INTRODUCTION

In this chapter we investigate a model of classical statistical physics in two dimensions, an spl(2,1) supersymmetric 15-vertex model, which is a generalization of the 6-vertex model. Each link in the square lattice can assume one of three states where two are bosonic and one is fermionic. The results for the spl(2,1) supersymmetric 15-vertex model are easily translated to the one-dimensional t-J model (for special values of the couplings t and J). Recently this model has attracted much interest in connection with high-T_c superconductivity. It describes a quantum system of electrons on a one dimensional chain, where at a lattice point there may be an electron with spin up or spin down or a hole. The hamiltonian for a lattice of L sites is given by [6]

$$\mathcal{H} = P\{-t\sum_{j,\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}P + J\sum_{j}(\vec{S}_j.\vec{S}_{j+1} - \frac{n_j n_{j+1}}{4}), \qquad (1.1)$$

where the projector $P = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy at one lattice point.

We present an explicit construction of the eigenvalues and eigenvectors of the transfer matrix of the spl(2,1) supersymmetric 15-vertex model using the algebraic nested Bethe ansatz method [17,42]. By this procedure the problem of finding the spectrum is reduced to the problem of solving a system of coupled transcendental equations, the

Bethe ansatz equations (BAE). We find three different kinds of BAE, which correspond to three different possible choices of pseudovacua. Two of these forms of BAE were already obtained by Lai [33], Schlottmann [34], Sutherland [35] and Sarkar [36] using similar methods. Moreover, we analyze in detail the algebraic structure of the eigenvectors obtained by this nested construction. From the invariance of the transfer matrix (and consequently of the one-dimensional t-J hamiltonian) with respect to the spl(2,1) superalgebra it follows that the eigenstates are classified in terms of supermultiplets corresponding to irreducible representations of this superalgebra. We analyze the structure of these representations. In addition, we prove that the Bethe ansatz states are highest weight vectors of the spl(2,1) superalgebra [37], which was investigated by Scheunert, Nahm and Rittenberg [43]. Therefore, by acting with the spl(2,1) lowering operators on the Bethe states we obtain additional eigenvectors. Finally, the total number of orthogonal eigenvectors generated by this procedure leads to a complete set of states. This result has been already announced in [38].

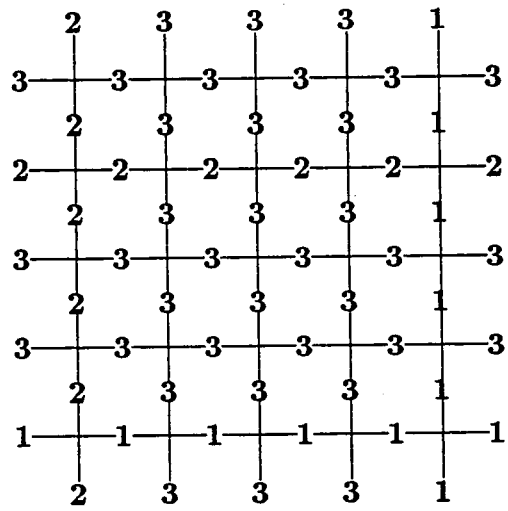
The chapter is organized as follows. In section 2 the spl(2,1) vertex model, as well as its transfer matrix, is defined on a two-dimensional lattice. We also give the relation between the transfer matrix and the one-dimensional supersymmetric t-J model. In section 3 we diagonalize the transfer matrix using the quantum inverse scattering method. In section 4 the algebraic structure of the Bethe vectors is investigated. Our results for lattices with small and large number of sites are illustrated in section 5, where the structure of the ground state and some low lying excitations are also discussed. In section 6 we give details of the proof of the completeness problem of the Bethe states for this model and section 7 contains a summary of the main results.

2. THE spl(2,1)-VERTEX MODEL AND YANG-BAXTER ALGEBRA

The graded 15-vertex model is a lattice model of classical statistical physics in two dimensions. Its partition function on a $L \times L'$ (L columns and L' rows) periodic square lattice is given as

$$\mathcal{Z} = \sum_{conf.} \prod_{x \in L \times L'} S(x), \tag{2.1}$$

where the sum extends over all allowed "bond configurations". Each bond can accept one of three states characterized by $\alpha = 1, 2, 3$, which can be bosonic (B) or fermionic (F). In what follows we will adopt the convention 1 = B, 2 = B, 3 = F. The figure below displays a possible configuration for a 5×5 square lattice with periodic boundary conditions



where each vertex weight $S(x)$ assumes one of the 15 allowed configurations at the lattice site x depicted below

$$\begin{array}{c} \alpha \\ \alpha - | - \alpha \\ \alpha \end{array} \equiv a \quad (\alpha \neq 3)$$

$$\begin{array}{c} 3 \\ 3 - | - 3 \\ 3 \end{array} \equiv w$$

$$\begin{array}{c} \beta \\ \alpha - | - \alpha \\ \beta \end{array} \equiv b \quad (\alpha \neq \beta)$$

$$\begin{array}{c} \beta \\ \alpha - | - \beta \\ \alpha \end{array} \equiv c \quad (\alpha \neq \beta)$$

and take the following values

$$S^{\gamma\delta}_{\alpha\beta}(v) = \gamma \overset{\delta}{-|-} \alpha \underset{\beta}{} = \sigma_{\gamma\delta}\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} - \frac{2}{v}\delta^{\gamma}_{\beta}\delta^{\delta}_{\alpha}. \tag{2.2}$$

The parametrization in terms of the spectral parameter "v" has been introduced for later convenience (see eq.(2.11)). The sign factor σ takes care of the statistics

$$\sigma_{\gamma\delta} = \begin{cases} -1, & \text{if } \gamma = \delta = 3 \text{ (fermionic)} \\ 1, & \text{otherwise .} \end{cases} \tag{2.3}$$

S can be considered as a matrix acting in the tensor product of two 3-dimensional auxiliary spaces $\mathbf{C}^3 \times \mathbf{C}^3$ and can be arranged as a 9×9 matrix.

$$S^{\gamma\delta}_{\alpha\beta}(v) = \left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & c & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{array}\right), \tag{2.4}$$

where

$$a = 1 - \frac{2}{v}, \quad b = 1, \quad c = -\frac{2}{v}, \quad w = -1 - \frac{2}{v}. \tag{2.5}$$

Let us now introduce the monodromy matrix $T(v)$, which is associated to a line of the lattice, say a horizontal line. Therefore, it can be written as the matrix product over the S's in the following way

$$T^{\gamma\{\delta\}}_{\alpha\{\beta\}}(v) = S^{\gamma\ \delta_1}_{\alpha_2\beta_1}(v) S^{\alpha_2\delta_2}_{\alpha_3\beta_2}(v) \ldots S^{\alpha_L\delta_L}_{\alpha\ \beta_L}(v)$$

$$\gamma - \overset{\{\delta\}}{\underset{\{\beta\}}{T}} - \alpha \ = \ \gamma \overset{\delta_1}{\underset{\beta_1}{-|-}} \overset{\delta_2}{\underset{\beta_2}{-|-}} \cdots \overset{\delta_L}{\underset{\beta_L}{-|-}} \alpha \ . \tag{2.6}$$

This monodromy matrix acts in the tensor product of an auxiliary space and a "quantum space" $\mathbf{C}^3 \times \mathbf{C}^{3L}$ and can be regarded as a 3×3 matrix of matrices acting in the "quantum

space"

$$T_{\alpha}^{\gamma}(v) = \begin{pmatrix} A & B_2 & B_3 \\ C_2 & D_1 & D_2 \\ C_3 & D_3 & D_4 \end{pmatrix}. \tag{2.7}$$

By using periodic boundary conditions, the transfer matrix is defined as a trace of the monodromy matrix in the auxiliary space

$$\tau_{\{\beta\}}^{\{\delta\}}(v) = \sum_{\alpha} \widetilde{T}_{\alpha\{\beta\}}^{\alpha\{\delta\}}(v) = \sum_{\alpha} \sigma_{\alpha\alpha}\sigma_{\alpha\{\delta\}} T_{\alpha\{\beta\}}^{\alpha\{\delta\}}(v), \tag{2.8}$$

where

$$\sigma_{\alpha\{\delta\}} = \prod_{i} \sigma_{\alpha\delta_i}. \tag{2.9}$$

Here the σ factors take into account the fact that we are dealing with bosons and fermions.

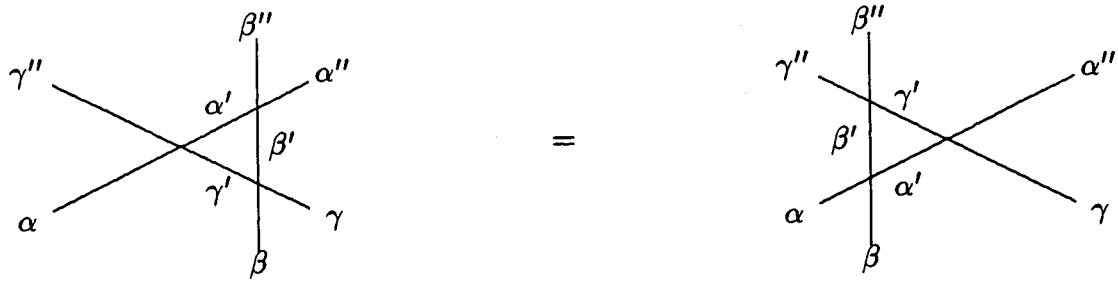
The thermodynamic properties of the vertex model can be obtained from the solutions of the eigenvalue problem of the transfer matrix

$$\tau\Psi = \lambda\Psi. \tag{2.10}$$

This eigenvalue problem will be solved in the next section by means of the nested Bethe ansatz.

It can easily be shown that the matrix S given by eq. (2.2) fulfills the Yang-Baxter equation

$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')S_{\alpha\ \gamma'}^{\alpha'\gamma''}(v)S_{\beta\ \gamma}^{\beta'\gamma'}(v') = S_{\beta'\gamma'}^{\beta''\gamma''}(v')S_{\alpha'\ \gamma}^{\alpha''\gamma'}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'). \tag{2.11}$$

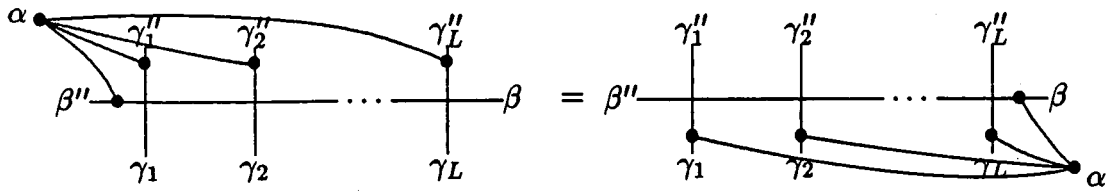


By means of iterations we can also prove the Yang-Baxter relation for the monodromy matrix T

$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')T_{\alpha\ \{\gamma'\}}^{\alpha'\{\gamma''\}}(v)T_{\beta\ \{\gamma\}}^{\beta'\{\gamma'\}}(v') = T_{\beta'\ \{\gamma'\}}^{\beta''\{\gamma''\}}(v')T_{\alpha'\ \{\gamma\}}^{\alpha''\{\gamma'\}}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'). \tag{2.12}$$

In addition conservation of fermions imply the following property of the T-matrix

$$\sigma_{\alpha\beta''}\sigma_{\alpha\{\gamma''\}}T_{\beta}^{\beta''\{\gamma''\}}{}_{\{\gamma\}}(v) = \sigma_{\alpha\beta}\sigma_{\alpha\{\gamma\}}T_{\beta}^{\beta''\{\gamma''\}}{}_{\{\gamma\}}(v), \tag{2.13}$$



for all $\alpha = 1, 2,$ or 3. Here σ counts the number of fermions and $\sigma_{\alpha\beta}$ is represented by a line connecting α and β.

The Yang-Baxter equation for the monodromy matrix (2.12) together with property (2.13) imply the commutativity of the transfer matrix for different spectral parameters

$$[\tau(v), \tau(v')] = 0. \tag{2.14}$$

This reflects the integrability of the model. In fact, the eigenvalue problem (2.10) can be solved exactly by the Bethe ansatz method.

At the end of this section we will show that the above defined transfer matrix is related to the 1-dimensional supersymmetric t-J model, such that if we solve the eigenvalue problem of the transfer matrix τ we will automatically diagonalize the hamiltonian of the 1-dimensional supersymmetric t-J model.

The hamiltonian of the t-J model for a one-dimensional lattice of L sites is given as [6]

$$\mathcal{H} = P\{-t\sum_{j,\sigma}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma} + c_{j+1,\sigma}^{\dagger}c_{j,\sigma})\}P + J\sum_{j}(\vec{S}_j.\vec{S}_{j+1} - \frac{n_j n_{j+1}}{4}), \tag{2.15}$$

where the $c_{j\pm}^{(\dagger)}$'s are spin up or down annihilation (creation) operators, the $\vec{S}_j$'s spin matrices and the n_j's occupation numbers of electrons at lattice site j. The projector $P = \prod_{j=1}^{L}(1 - n_{j\uparrow}n_{j\downarrow})$ restricts the Hilbert space by the constraint of no double occupancy at one lattice point. Therefore, at each lattice site we have three possibilities $(1,2,3) \equiv (\uparrow, \downarrow, 0)$, i.e. an electron with spin up or down or no electron (hole). This hamiltonian can be rewritten in terms of Hubbard's projection operators [44]

$$X_j^{\alpha\beta} = |\alpha_j\rangle\langle\beta_j|; \qquad \alpha, \beta = 1, 2, 3\ , \tag{2.16}$$

where $|1_j(2_j)\rangle$ denotes an electron with spin up (down) and $|3_j\rangle$ a hole at site j. Using (2.16), up to a chemical potential the hamiltonian reads

$$\mathcal{H} = -t\sum_{\alpha=1}^{2}\sum_{j=1}^{L}(X_j^{\alpha 3}X_{j+1}^{3\alpha} + X_{j+1}^{\alpha 3}X_j^{3\alpha}) + \frac{J}{2}\sum_{j=1}^{L}\left(\sum_{\alpha,\beta=1}^{2} X_j^{\alpha\beta}X_{j+1}^{\beta\alpha} - X_j^{33}X_{j+1}^{33}\right) \quad (2.17)$$

For convenience we will consider the hole operators as fermions and the spin operators as bosons. In fact, this choice is possible since in 1-dimension there exists a transformation exchanging bosons and fermions. Therefore the spectrum of the t-J model with two fermions and and one boson is equivalent to the spectrum of the t-J model with two bosons and one fermion (for even L) [36].

For $J = 2t$ the t-J model is "supersymmetric" [45] and connected to the previously defined vertex model through the relation

$$\mathcal{H} = -2\frac{\partial}{\partial v}\ln(v^L\tau(v))|_{v=0}. \quad (2.18)$$

The proof of this identity is analogous to the one for the isotropic Heisenberg model [9,11].

3. CONSTRUCTION OF BETHE EIGENVECTORS

The main subject of this section will be solving the eigenvalue problem of the transfer matrix

$$\tau\Psi = \lambda\Psi \quad (3.1)$$

through an algebraic construction [17] based on the Yang-Baxter algebra of the monodromy matrices

$$S_{\alpha'\beta'}^{\alpha''\beta''}(v-v')T_{\alpha\ \{\gamma'\}}^{\alpha'\{\gamma''\}}(v)T_{\beta\ \{\gamma\}}^{\beta'\{\gamma'\}}(v') = T_{\beta'\ \{\gamma'\}}^{\beta''\{\gamma''\}}(v')T_{\alpha'\ \{\gamma\}}^{\alpha''\{\gamma'\}}(v)S_{\alpha\ \beta}^{\alpha'\beta'}(v-v'). \quad (3.2)$$

The monodromy matrix T can be written as a 3x3 matrix

$$\left(\begin{array}{c|cc} A & B_2 & B_3 \\ \hline C_2 & D_1 & D_2 \\ C_3 & D_3 & D_4 \end{array}\right). \quad (3.3)$$

This suggests solving the problem by means of the nested Bethe ansatz with two levels [42]. The transfer matrix is given by a trace of the monodromy matrix T (see eq.

(2.8)). For the first level Bethe ansatz the operators $B_\alpha(C_\alpha)$ ($\alpha = 2,3$) play the role of creation (annihilation) operators of "pseudoparticles". The first level "pseudovacuum" Φ is defined by the equation

$$C^{\{\beta\}}_{\gamma\{\beta'\}}\Phi^{\{\beta'\}} = 0$$

$$\gamma - \begin{matrix}\beta_1 & \beta_2 & & \beta_L \\ | & | & & | \\ \beta'_1 & \beta'_2 & & \beta'_L\end{matrix} - 1 \;\Phi^{\{\beta'\}} = 0 \qquad \text{for} \quad \gamma = 2,3. \tag{3.4}$$

Since at a vertex a generalized "ice rule" holds (see eq. (2.2)) the solution of this equation is

$$\Phi^{\{\beta\}} = \prod_{i=1}^{L}\delta_{\beta_i 1} = \begin{matrix}\beta_1 & \beta_2 & & \beta_L \\ | & | & \cdots & | \\ 1 & 1 & & 1\end{matrix} \quad . \tag{3.5}$$

This pseudovacuum is an eigenstate of A

$$A^{\{\beta\}}_{\{\beta'\}}(v)\Phi^{\{\beta'\}} = a^L(v)\Phi^{\{\beta\}}$$

$$1 - \begin{matrix}\beta_1 & \beta_2 & & \beta_L \\ | & | & & | \\ 1 & 1 & & 1\end{matrix} - 1 = 1 - \begin{matrix}1 & 1 & & 1 \\ | & | & & | \\ 1 & 1 & & 1\end{matrix} - 1 \tag{3.6}$$

and also of D_1 and D_4

$$D_{1(4)\{\beta'\}}^{\{\beta\}}(v)\Phi^{\{\beta'\}} = b^L(v)\Phi^{\{\beta\}}$$

$$\alpha - \begin{matrix}\beta_1 & \beta_2 & & \beta_L \\ | & | & & | \\ 1 & 1 & & 1\end{matrix} - \alpha = \alpha - \begin{matrix}1 & 1 & & 1 \\ | & | & & | \\ 1 & 1 & & 1\end{matrix} - \alpha \tag{3.7}$$

($\alpha = 2$ and 3, respectively). Because of the special form of the matrix S of eq. (2.2) the summations over the internal lines in eqs. (3.6) and (3.7) are trivial. In eq. (3.6) they can assume only the value 1, and in eq. (3.7) only the fixed value $\alpha = 2$ or 3, respectively. The action of B_α ($\alpha = 2$ or 3) on the "pseudovacuum" yields new states. So, the $\{B_\alpha\}$ can be considered as "creation operators" and the eigenvector of the transfer matrix can be obtained by successive application of the B's according to the first level Bethe ansatz

$$\Psi^{\{\beta\}} = B_{\alpha_1\{\eta_1\}}^{\{\beta\}}(v_1)B_{\alpha_2\{\eta_2\}}^{\{\eta_1\}}(v_2)\ldots B_{\alpha_N\{\eta_N\}}^{\{\eta_{N-1}\}}(v_N)\Phi^{\{\eta_N\}}\Psi_{(1)}^{\{\alpha\}}, \tag{3.8}$$

where the summations over the α_i ($i = 1,\ldots,N$) are restricted to $\alpha_i = 2,\ 3$. The coefficients $\Psi_{(1)}^{\{\alpha\}}$ are to be to be determined by the second level Bethe ansatz. This

means the eigenvalue problem of the transfer matrix (3.1) will be solved in a recurrent way (nested Bethe ansatz method). The requirement that Ψ is an eigenvector of τ leads to another eigenvalue problem for a new transfer matrix $\tau_{(1)}$, as will be shown later. Now we start to solve eq. (3.1). Following the general strategy of the algebraic Bethe ansatz [17] we apply the transfer matrix $\tau(v)$ (2.8) to the state Ψ given by eq.(3.8).

$$\tau^{\{\beta''\}}_{\{\beta'\}}(v)\Psi^{\{\beta'\}} = \left(A^{\{\beta''\}}_{\{\beta'\}}(v) + \tau_D{}^{\{\beta''\}}_{\{\beta'\}}(v)\right)\Psi^{\{\beta'\}}, \tag{3.9}$$

where

$$\tau_D{}^{\{\beta''\}}_{\{\beta'\}}(v) = \sum_{\alpha=2}^{3} \widetilde{T}^{\alpha\{\beta''\}}_{\alpha\{\beta'\}}(v) = \sum_{\alpha=2}^{3} \sigma_{\alpha\alpha}\sigma_{\alpha\{\beta''\}} T^{\alpha\{\beta''\}}_{\alpha\{\beta'\}}(v). \tag{3.10}$$

In order to commute $A(v)$, $D_1(v)$ and $D_4(v)$ through all $B(v_i)$ towards Φ and then apply (3.6) and (3.7) we use the property (2.13) and the following commutation rules, derived from the Yang-Baxter relation (3.2)

$$A(v)B_\alpha(v') = \frac{a(v'-v)}{b(v'-v)}B_\alpha(v')A(v) - \frac{c(v'-v)}{b(v'-v)}B_\alpha(v)A(v') \tag{3.11}$$

and

$$T^\gamma_{\gamma'}(v)B_\alpha(v') = \frac{1}{b(v-v')}\left(B_{\alpha'}(v')T^\gamma_{\gamma''}(v)S^{\gamma''\alpha'}_{\gamma'\alpha}(v-v') - c(v-v')B_{\gamma'}(v)T^\gamma_\alpha(v')\right) \tag{3.12}$$

$$B_\alpha(v)B_\beta(v') = \frac{1}{a(v-v')}B_{\beta'}(v')B_{\alpha'}(v)S^{\alpha'\beta'}_{\alpha\beta}(v-v'). \tag{3.13}$$

All indices of the auxiliary space in eqs. (3.11), (3.12) and (3.13) assume only the values 2 and 3. Using eq.(3.11) two types of terms arise when A is commuted through B_α. In the first type A and B_α preserve their arguments and in the second type their arguments are exchanged. The first kind of terms are called "wanted terms", since they will give a vector proportional to Ψ and the second type are the "unwanted terms (u.t.)". Then, using (3.9), (3.8), (2.13), (3.11) and (3.6), we get

$$A^{\{\beta''\}}_{\{\beta'\}}(v)\Psi^{\{\beta'\}} = \lambda_A(v)\Psi^{\{\beta''\}} + u.t.(A), \tag{3.14}$$

where the coefficient λ_A is given by

$$\lambda_A(v) = a^L(v)\prod_{i=1}^{N}\frac{a(v_i-v)}{b(v_i-v)}. \tag{3.15}$$

Correspondingly we obtain from (3.12) the wanted and unwanted terms in the form

$$\tau_{D\{\beta'\}}^{\{\beta''\}}(v)\Psi^{\{\beta'\}} = b^L(v)\prod_{i=1}^{N}\frac{1}{b(v-v_i)}B_{\alpha_1'\{\eta_1\}}^{\{\beta''\}}(v_1)B_{\alpha_2'\{\eta_2\}}^{\{\eta_1\}}(v_2)\dots \\ \dots B_{\alpha_N'\{\eta_N\}}^{\{\eta_{N-1}\}}(v_N)\Phi^{\{\eta_N\}}\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v,\{v_i\})\Psi_{(1)}^{\{\alpha\}} + u.t.(D), \tag{3.16}$$

where we have introduced a new (the second level) transfer matrix

$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v,\{v_i\}) = \sum_{\beta=2}^{3}\sigma_{\beta\beta}\sigma_{\beta\{\alpha'\}}T_{(1)\beta\{\alpha\}}^{\beta\{\alpha'\}}(v,\{v_i\}) \tag{3.17}$$

as a trace (over only $\beta = 2$ and 3) of the second level monodromy matrix. This is given by $T_{(1)} = S(v-v_N)\dots S(v-v_1)$ in analogy to eq. (2.6). Now, however, all indices (the external and the internal ones) assume only the values 2 and 3, as in the internal block of the matrix T denoted in eq. (3.3). In order to obtain in eq. (3.16) a "wanted term" proportional to Ψ, the vector $\Psi_{(1)}$ has to fulfil the eigenvalue equation

$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v,\{v_i\})\Psi_{(1)}^{\{\alpha\}} = \lambda_{(1)}(v,\{v_i\})\Psi_{(1)}^{\{\alpha'\}}. \tag{3.18}$$

which is solved by the second level Bethe ansatz. The monodromy matrix $T_{(1)}$ belongs to an SL(1,1) 6-vertex model slightly modified compared to the $SU(2)$ one due to the presence of fermions. If we identify $T_{(1)}{}_2^2 \equiv A_{(1)}$, $T_{(1)}{}_3^2 \equiv B_{(1)}$, $T_{(1)}{}_2^3 \equiv C_{(1)}$ and $T_{(1)}{}_3^3 \equiv D_{(1)}$ again $B_{(1)}$ ($C_{(1)}$) can be interpreted as a creation (annihilation) operator with respect to the "pseudovacuum" $\Phi_{(1)}$, which is now of the form

$$\Phi_{(1)}^{\{\alpha\}} = \prod_{i=1}^{N}\delta_{\alpha_i,2} = \begin{matrix}\alpha_N & & \alpha_2 & \alpha_1\\ | & \cdots & | & |\\ 2 & & 2 & 2\end{matrix}. \tag{3.19}$$

It is an eigenstate of $A_{(1)}$ and $D_{(1)}$, satisfying

$$A_{(1)\{\alpha'\}}^{\{\alpha\}}(v,\{v_i\})\Phi_{(1)}^{\{\alpha'\}} = \prod_{i=1}^{N}a(v-v_i)\Phi_{(1)}^{\{\alpha\}} \\ 2\,\begin{matrix}\alpha_N & & \alpha_2 & \alpha_1\\ | & \cdots & | & |\\ 2 & & 2 & 2\end{matrix}\,2 = 2\,\begin{matrix}2 & 2 & & 2\\ | & | & \cdots & |\\ 2 & 2 & & 2\end{matrix}\,2 \tag{3.20}$$

$$D_{(1)\{\alpha'\}}^{\{\alpha\}}(v,\{v_i\})\Phi_{(1)}^{\{\alpha'\}} = \prod_{i=1}^{N}b(v-v_i)\Phi_{(1)}^{\{\alpha\}} \\ 3\,\begin{matrix}\alpha_N & & \alpha_2 & \alpha_1\\ | & \cdots & | & |\\ 2 & & 2 & 2\end{matrix}\,3 = 3\,\begin{matrix}2 & 2 & & 2\\ | & | & \cdots & |\\ 2 & 2 & & 2\end{matrix}\,3. \tag{3.21}$$

The summations over the internal lines in eqs. (3.20) and (3.21) are only over the values 2 and 3, respectively. The eigenvector $\Psi_{(1)}$ of $\tau_{(1)}$ is given by the second level Bethe ansatz

$$\Psi_{(1)}^{\{\alpha\}} = B_{(1)\{\eta_1\}}^{\{\alpha\}}(\gamma_1,\{v_i\})B_{(1)\{\eta_2\}}^{\{\eta_1\}}(\gamma_2,\{v_i\})\dots B_{(1)\{\eta_M\}}^{\{\eta_{M-1}\}}(\gamma_M,\{v_i\})\Phi_{(1)}^{\{\eta_M\}}. \tag{3.22}$$

Following a strategy analogous to the one above, we apply $\tau_{(1)}$ to the state $\Psi_{(1)}$ and commute $A_{(1)}(v,\{v_i\})$ and $D_{(1)}(v,\{v_i\})$ through the $B_{(1)}(\gamma_\alpha,\{v_i\})$ towards $\Phi_{(1)}$ and then use (3.20) and (3.21). Since the Yang-Baxter algebra for the monodromy matrices (3.2) is also valid in the inhomogeneous cases when $T(v)$ is replaced by $T(v,\{v_i\})$ [46], we derive the following commutation relations

$$\begin{aligned} A_{(1)}(v,\{v_i\})B_{(1)}(v',\{v_i\}) =& \frac{a(v'-v)}{b(v'-v)}B_{(1)}(v',\{v_i\})A_{(1)}(v,\{v_i\}) \\ &- \frac{c(v'-v)}{b(v'-v)}B_{(1)}(v,\{v_i\})A_{(1)}(v',\{v_i\}) \ , \end{aligned} \tag{3.23}$$

$$\begin{aligned} D_{(1)}(v,\{v_i\})B_{(1)}(v',\{v_i\}) =& \frac{w(v-v')}{b(v-v')}B_{(1)}(v',\{v_i\})D_{(1)}(v,\{v_i\}) \\ &- \frac{c(v-v')}{b(v-v')}B_{(1)}(v,\{v_i\})D_{(1)}(v',\{v_i\}) \ , \end{aligned} \tag{3.24}$$

$$B_{(1)}(v,\{v\})B_{(1)}(v',\{v\}) = \frac{w(v-v')}{a(v-v')}B_{(1)}(v',\{v\})B_{(1)}(v,\{v\}). \tag{3.25}$$

Using eqs. (3.17), (3.22), (2.13), (3.23), (3.24), (3.20) and (3.21) as above we obtain again wanted and unwanted terms

$$\tau_{(1)\{\alpha\}}^{\{\alpha'\}}(v,\{v_i\})\Psi_{(1)}^{\{\alpha\}} = \big(\lambda_{A_{(1)}}(v,\{v_i\}) + \lambda_{D_{(1)}}(v,\{v_i\})\big)\Psi_{(1)}^{\{\alpha'\}} + u.t.(A_{(1)}) + u.t.(D_{(1)}), \tag{3.26}$$

where

$$\lambda_{A_{(1)}} = \prod_{i=1}^{N} a(v-v_i)\prod_{\beta=1}^{M}\frac{a(\gamma_\beta - v)}{b(\gamma_\beta - v)} \ , \tag{3.27}$$

$$\lambda_{D_{(1)}} = -(-1)^M\prod_{i=1}^{N} b(v-v_i)\prod_{\beta=1}^{M}\frac{w(v-\gamma_\beta)}{b(v-\gamma_\beta)} \ . \tag{3.28}$$

Substituting these equations in (3.16) and taking (3.8) into account we get, in case the unwanted terms $u.t._{(A_{(1)})}$ and $u.t._{(D_{(1)})}$ cancel,

$$\tau_{D\{\beta'\}}^{\{\beta''\}}(v)\Psi^{\{\beta'\}} = \big(\lambda_{D_I}(v) + \lambda_{D_{II}}(v)\big)\Psi^{\{\beta''\}} + u.t.(D), \tag{3.29}$$

where λ_{D_I} and $\lambda_{D_{II}}$ are given by

$$\lambda_{D_I} = b^L(v) \prod_{i=1}^{N} \frac{a(v-v_i)}{b(v-v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - v)}{b(\gamma_\beta - v)} , \tag{3.30}$$

$$\lambda_{D_{II}} = -(-1)^M b^L(v) \prod_{\beta=1}^{M} \frac{w(v-\gamma_\beta)}{b(v-\gamma_\beta)} . \tag{3.31}$$

Finally, combining eqs. (3.14) and (3.29) we have, again if the unwanted terms $u.t.(A)$ and $u.t.(D)$ cancel

$$\tau^{\{\beta''\}}_{\{\beta'\}}(v)\Psi^{\{\beta'\}} = \lambda(v)\Psi^{\{\beta''\}}, \tag{3.32}$$

where

$$\lambda(v) = \lambda_A(v) + \lambda_{D_I}(v) + \lambda_{D_{II}}(v). \tag{3.33}$$

The cancellation of all unwanted terms ensure that Ψ, as given by eq. (3.8), is an eigenstate of the transfer matrix τ (2.8) with eigenvalue $\lambda(v)$ of eq. (3.33).

In the appendix A we show that the unwanted terms indeed vanish if the Bethe ansatz equations hold

$$\left(\frac{a(v_j)}{b(v_j)}\right)^L \prod_{i=1}^{N} \frac{a(v_i-v_j)}{b(v_i-v_j)} \frac{b(v_j-v_i)}{a(v_j-v_i)} \prod_{\beta=1}^{M} \frac{b(\gamma_\beta - v_j)}{a(\gamma_\beta - v_j)} = -1 , \quad j = 1,\ldots,N, \tag{3.34}$$

$$(-1)^M \prod_{i=1}^{N} \frac{a(\gamma_\alpha - v_i)}{b(\gamma_\alpha - v_i)} \prod_{\beta=1}^{M} \frac{a(\gamma_\beta - \gamma_\alpha)}{b(\gamma_\beta - \gamma_\alpha)} \frac{b(\gamma_\alpha - \gamma_\beta)}{w(\gamma_\alpha - \gamma_\beta)} = 1 , \quad \alpha = 1,\ldots M, \tag{3.35}$$

where N is the number of holes plus down spins and M is the number of holes. Another way to obtain these equations is to require that the eigenvalue $\lambda(v)$ (3.33) has no poles at $v = v_i$ and $v = \gamma_\beta$. Using (2.5) and making the change of variables $v \to iv+1$, $\gamma \to i\gamma + 2$ we obtain

$$\left(\frac{v_j+i}{v_j-i}\right)^L = \prod_{k\neq j}^{N} \frac{v_j - v_k + 2i}{v_j - v_k - 2i} \prod_{\beta=1}^{M} \frac{v_j - \gamma_\beta - i}{v_j - \gamma_\beta + i} , \quad j = 1,\ldots,N, \tag{3.36}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha - v_j + i}{\gamma_\alpha - v_j - i} = 1 , \quad \alpha = 1,\ldots,M. \tag{3.37}$$

This form of the Bethe ansatz equations (BAE) was previously derived by Sutherland [35] and later by Sarkar using a generalized permutation operator [36]. We stress that

this procedure could be repeated with two other choices of the pseudovacuum leading to two other forms of the BAE. The pseudovacua of both levels of the Bethe ansatz Φ and $\Phi_{(1)}$ (see eqs. (3.5) and (3.19)), which we used above, consist of states of kind 1=B and 2=B, respectively. Basically, the change of pseudovacuum is determined by altering the initial convention (1=B, 2=B, 3=F). Using (1=F, 2=B, 3=B) we get

$$\left(\frac{v_j+i}{v_j-i}\right)^L = \prod_{\beta=1}^{M} \frac{v_j-\gamma_\beta+i}{v_j-\gamma_\beta-i}\,, \quad j=1,\ldots,N, \tag{3.38}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha-v_j-i}{\gamma_\alpha-v_j+i} = \prod_{\beta=1}^{M} \frac{\gamma_\alpha-\gamma_\beta-2i}{\gamma_\alpha-\gamma_\beta+2i}\,, \quad \alpha=1,\ldots M, \tag{3.39}$$

where N is the total number of spins and M is the number of spins down. These equations were already obtained by Lai [33] and Schlottmann [34] using the coordinate Bethe ansatz method.

Finally, the choice (1=B, 2=F, 3=B) leads to a new form of the BAE

$$\left(\frac{v_j-i}{v_j+i}\right)^L = \prod_{\beta=1}^{M} \frac{v_j-\gamma_\beta-i}{v_j-\gamma_\beta+i}\,, \quad j=1,\ldots,N, \tag{3.40}$$

$$\prod_{j=1}^{N} \frac{\gamma_\alpha-v_j+i}{\gamma_\alpha-v_j-i} = 1\,, \quad \alpha=1,\ldots M, \tag{3.41}$$

where N is the number of holes plus spin downs and M is the number of spins down. In the following we will work with the BAE's (3.36) and (3.37), since this is the most convenient form for the present investigation.

We have reduced the eigenvalue problem of the transfer matrix (3.1) to a system of coupled algebraic equations for the parameters $\{v_j\}$ $(j=1,\ldots,N)$ and $\{\gamma_\alpha\}$ $(\alpha=1,\ldots,M)$. The basic procedure to solve eqs. (3.36) and (3.37) is to adopt the string-conjecture, which means that the v's appear as strings and all roots γ's are real

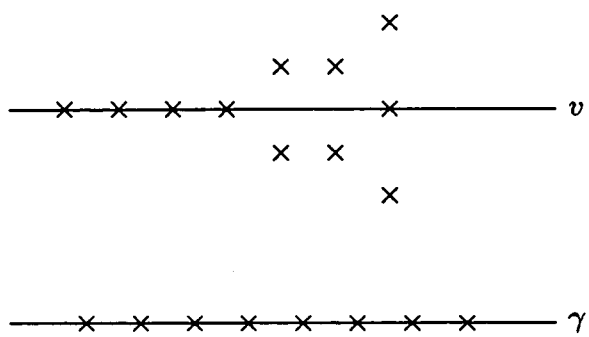
$$\begin{aligned} &v_{\alpha j}^n = v_\alpha^n + i(n+1-2j); \qquad j=1,\ldots,n; \quad \alpha=1,\ldots,N_n; \quad n=1,2,\ldots \\ &\gamma_\beta = \text{real}; \qquad \beta=1,\ldots,M \end{aligned} \tag{3.42}$$

where v_α^n is the position of the center of the string on the real v-axis. The number of n-strings N_n satisfy the relation

$$N = \sum_n n N_n. \tag{3.43}$$

This hypothesis for the v's can be easily understood by heuristic arguments, analogously to the isotropic Heisenberg model [23,47]. To understand absence of complex roots for the γ's we apply the following argument, which is similar to that one developed by Takahashi for the one dimensional electron gas with a repulsive delta-function [48]. If all v_i are real or appear as complex conjugate pairs, $\mathrm{Im}\,\gamma_\alpha > 0$ implies that the absolute value of the left hand side of eq.(3.37) is larger than unity. Therefore, $\mathrm{Im}\,\gamma_\alpha > 0$ is not possible. In the same way we can prove that $\mathrm{Im}\,\gamma_\alpha < 0$ is not possible.

We can see here the great advantage of using this form of BAE. In the other two forms not only the parameters v's but also the roots γ's appear as strings. This means that counting the states is much more complicated. Although we are not able to prove the string-conjecture rigorously, we will assume it to be valid. Since Bethe [15], assumptions of this kind have been widely used by many authors ([47] and references therein). The string conjecture is illustrated below for a particular case ($N_1 = 4$, $N_2 = 2$, $N_3 = 1$) $N = 11$ and $M = 8$.



Applying the string conjecture (3.42) in (3.36) and (3.37) and taking the product over j in (3.36) we obtain

$$\underbrace{\prod_j^n \left(\frac{v_\alpha^n + i(n-2j+2)}{v_\alpha^n + i(n-2j)}\right)^L}_{} = \prod_m^{N_m}\prod_\beta \underbrace{\prod_k^m \prod_j^n \frac{v_\alpha^n - v_\beta^m + i(n-m-2j+2k+2)}{v_\alpha^n - v_\beta^m + i(n-m-2j+2k-2)}}_{} \times \prod_\xi^M \underbrace{\prod_j^n \frac{v_\alpha^n - \gamma_\xi + i(n-2j)}{v_\alpha^n - \gamma_\xi + i(n-2j+2)}}_{}$$

$$\prod_n^{N_n}\prod_\alpha \underbrace{\prod_j^n \frac{v_\alpha^n - \gamma_\beta + i(n-2j)}{v_\alpha^n - \gamma_\beta + i(n-2j+2)}}_{} = 1$$

The products indicated above can be performed resulting in

$$e^L\left(\frac{v_\alpha^n}{n}\right) = \prod_m \prod_\beta^{N_m} E_{nm}(v_\alpha^n - v_\beta^m) \prod_\xi^M e^{-1}\left(\frac{v_\alpha^n - \gamma_\xi}{n}\right)$$

$$\prod_n \prod_\alpha^{N_n} e^{-1}\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = \quad 1$$

where $e(x) = \frac{x+i}{x-i}$ and

$$E_{nm}(x) = \begin{cases} e\left(\frac{x}{|n-m|}\right) + e^2\left(\frac{x}{|n-m|+2}\right) + \ldots + e^2\left(\frac{x}{n+m-2}\right) + e\left(\frac{x}{n+m}\right) & \text{for } n \neq m \\ e^2\left(\frac{x}{2}\right) + e^2\left(\frac{x}{4}\right) + \ldots + e^2\left(\frac{x}{2n-2}\right) + e\left(\frac{x}{2n}\right) & \text{for } n = m. \end{cases}$$

Taking the logarithm of these equations and using $\ln e(x) = \frac{2}{i}\arctan x - \ln(-1)$ we get

$$L\theta\left(\frac{v_\alpha^n}{n}\right) - \underbrace{\sum_m \sum_{\beta=1}^{N_m}}_{(m,\beta)\neq(n,\alpha)} \Theta_{nm}(v_\alpha^n - v_\beta^m) + \sum_{\beta=1}^{M} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi I_\alpha^n, \tag{3.44}$$

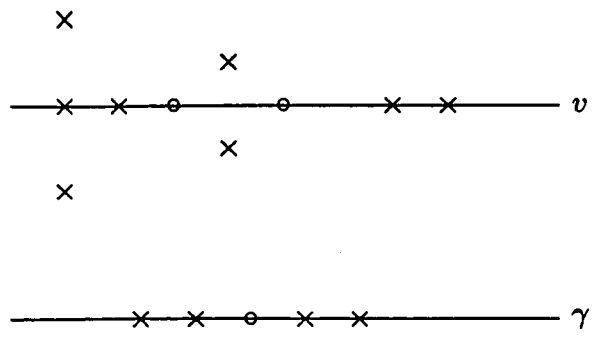
$$\sum_n \sum_{\alpha=1}^{N_n} \theta\left(\frac{v_\alpha^n - \gamma_\beta}{n}\right) = 2\pi J_\beta, \tag{3.45}$$

where $\theta(x) = 2\arctan x$ and

$$\Theta_{nm}(x) = \begin{cases} \theta\left(\frac{x}{|n-m|}\right) + 2\theta\left(\frac{x}{|n-m|+2}\right) + \ldots + 2\theta\left(\frac{x}{n+m-2}\right) + \theta\left(\frac{x}{n+m}\right) & \text{for } n \neq m \\ 2\theta\left(\frac{x}{2}\right) + 2\theta\left(\frac{x}{4}\right) + \ldots + 2\theta\left(\frac{x}{2n-2}\right) + \theta\left(\frac{x}{2n}\right) & \text{for } n = m. \end{cases} \tag{3.46}$$

Hence the solutions of (3.36) and (3.37) are parametrized in terms of the numbers I_α^n and J_β. Here, the I_α^n are integers (half-integers) if $L + M - N_n$ is odd (even) and the J_β are integers (half-integers) if $\sum_n N_n$ is even (odd). We assume that for every n the numbers I_α, $\alpha = 1 \ldots N_n$ are ordered such that $I_1 < I_2 < \ldots I_{N_n}$ (the same for $\{J_\beta\}$). The admissible values of the numbers I_α^n and J_β are denoted by "vacancies". A "BA-hole" corresponds to a non-occupied place in the set of numbers $\{I^n\}$ or $\{J\}$ for a solution of the BAE (unfilled vacancy). A typical set of roots of the Bethe ansatz equations in the complex plane, including real roots, BA-holes and n-strings is displayed

below



The numbers I^n_α and J_β are limited to the intervals

$$|I^n_\alpha| \leq I^n_{\max} = \frac{1}{2}(L + M - \sum_m t_{nm} N_m - 1), \tag{3.47}$$

$$|J_\beta| \leq J_{\max} = \frac{1}{2}(\sum_n N_n - 2), \tag{3.48}$$

where $t_{nm} = 2\min(n,m) - \delta_{nm}$. In fact, all sets $\{I^n_\alpha, J_\beta\}$ where the I's and J's are pairwise different specify all the Bethe vectors ($|\psi_{\text{Bethe}}\rangle_{N,M}$). They are highest weight vectors of an spl(2,1) superalgebra, as we will show in the next section.

In order to avoid misunderstandings we should add some general remarks on the string conjecture (3.42) and the bounds $I^n_{\max}$ and $J_{\max}$ given by eqs. (3.47) and (3.48). Both statements are to be considered as assumptions, they cannot be proven rigorously. In fact they are not exact. There are finite size corrections of the string configurations of order $O(e^{-L})$ for fixed string centers v^n_α and of order $O(1)$ near to the boundary $v^n_{\max}$ (given by $I^n_{\max}$), producing "exotic solutions". On the other hand a naive estimate of $I^n_{\max}$ from eq. (3.44) would suggest additional solutions (for $n \geq 2$) which are canceled by assumption (3.47). However, both assumptions together lead to the correct number of states, as is well known for the SU(2) case [23,47] and will be proven below for the spl(2,1) case. Obviously, the effects of the two phenomena mentioned above compensate for this computation. In addition to the "exotic solutions" mentioned above, there exist also "wide pairs" and "quartets" [49] if the density of real roots is large enough. It is believed that these problems may be avoided and exotic effects may be neglected, if one considers the following thermodynamic limit. Introduce a symmetry breaking magnetic field B and take first the limit $L \to \infty$ and then $B \to 0$. It should be stressed that many features of the Bethe ansatz are not well understood.

At the end of this section we apply the results obtained for the spl(2,1) vertex model to the supersymmetric t-J model. Using the identity (2.18) it is possible to obtain the energy eigenvalues of the t-J model from the eigenvalues of the transfer matrix (3.33). The terms $\lambda_{D_I,D_{II}}$ given by eqs. (3.30) and (3.31) do not contribute and from eq. (3.15) we find

$$E = L - \sum_{j=1}^{N} \frac{4}{1+{v_j}^2}. \tag{3.49}$$

Using the string-conjecture (3.42) the above equation can be arranged as follows

$$E = L - \sum_{n} \sum_{\alpha=1}^{N_n} \frac{4n}{n^2+v_\alpha^{n2}}. \tag{3.50}$$

For later convenience we define other important physical quantities as the momentum [23]

$$P = -i \sum_{j=1}^{N} \ln\left(\frac{v_j+i}{v_j-i}\right) = \sum_{n} \sum_{\alpha=1}^{N_n} (-2\arctan v_\alpha^n + \pi), \tag{3.51},$$

the magnetization S_z and the number of electrons (see eq. (4.10))

$$S_z = \frac{1}{2}(n_\uparrow - n_\downarrow) = \frac{1}{2}(L - 2N + M), \tag{3.52}$$

$$Q = n_\uparrow + n_\downarrow = L - M. \tag{3.53}$$

4. ALGEBRAIC PROPERTIES OF THE BETHE STATES

In this section we analyze the algebraic properties of the Bethe states. By asymptotic expansion ($v \to \infty$) we define a matrix $\tilde{M}$ of operators in the "quantum space" as follows

$$T_{\alpha}^{\alpha''}{}_{\{\gamma\}}^{\{\gamma''\}}(v) = \sigma_{\alpha''\{\gamma''\}}\delta_{\alpha}^{\alpha''}\delta_{\{\gamma\}}^{\{\gamma''\}} - \frac{2}{v}\sigma_{\alpha''\alpha}\sigma_{\alpha''\{\gamma''\}}\tilde{M}_{\alpha}^{\alpha''}{}_{\{\gamma\}}^{\{\gamma''\}} + O(v^{-2}). \tag{4.1}$$

It turns out that the entries of $\tilde{M}$ are generators of the superalgebra $spl(2,1)$. We prove the commutation relations using the Yang-Baxter relation (3.2) for the monodromy matrix and the property (2.13). For $v \to \infty$ we have (in what follows we will omit the quantum space indices and write them only whenever necessary)

$$\tilde{M}_{\alpha}^{\alpha''}\tilde{T}_{\beta}^{\beta''}(v') - \Sigma(\alpha'',\beta'',\alpha,\beta)\tilde{T}_{\beta}^{\beta''}(v')\tilde{M}_{\alpha}^{\alpha''} = \tilde{T}_{\alpha}^{\beta''}(v')\delta_{\beta}^{\alpha''} - \Sigma(\alpha'',\beta'',\alpha,\beta)\delta_{\alpha}^{\beta''}\tilde{T}_{\beta}^{\alpha''}(v'). \tag{4.2}$$

Here the sign function Σ is given by

$$\Sigma(\alpha'',\beta'',\alpha,\beta) = \sigma_{\alpha''\beta''}\sigma_{\alpha''\beta}\sigma_{\alpha\beta''}\sigma_{\alpha\beta}. \tag{4.3}$$

Furthermore, taking $v' \to \infty$ we get

$$\tilde{M}_{\alpha}^{\alpha''}\tilde{M}_{\beta}^{\beta''} - \Sigma(\alpha'',\beta'',\alpha,\beta)\tilde{M}_{\beta}^{\beta''}\tilde{M}_{\alpha}^{\alpha''} = \tilde{M}_{\alpha}^{\beta''}\delta_{\beta}^{\alpha''} - \Sigma(\alpha'',\beta'',\alpha,\beta)\delta_{\alpha}^{\beta''}\tilde{M}_{\beta}^{\alpha''}. \tag{4.4}$$

This relation represents the commutation and anti-commutation rules of the spl(2,1) superalgebra [43]. The generators $\tilde{M}_3^{\alpha}, \tilde{M}_{\alpha}^3$, $(\alpha \neq 3)$ are fermionic, whereas the $\tilde{M}_3^3$ and $\tilde{M}_{\beta}^{\alpha}$ $(\alpha,\beta \neq 3)$ are bosonic. The sign factors Σ take into account the statistics, i.e., $\Sigma = -1(1)$ if we are dealing with odd (even) generators. Eq. (4.4) can be written in the compact form

$$[\tilde{M}_{\alpha}^{\alpha''}, \tilde{M}_{\beta}^{\beta''}]_{\pm} = \tilde{M}_{\alpha}^{\beta''}\delta_{\beta}^{\alpha''} \pm \delta_{\alpha}^{\beta''}\tilde{M}_{\beta}^{\alpha''}. \tag{4.5}$$

In addition, from (4.2) it is easy to see that the transfer matrix τ (2.8) is invariant with respect to the spl(2,1) superalgebra, i.e.

$$[\tilde{M}_{\alpha}^{\alpha''}, \tau(v')] = 0. \tag{4.6}$$

Notice that the results (4.2), (4.4), (4.5) and (4.6) are also valid if we change the convention ($1 = B$, $2 = B$, $3 = F$). The position of the fermion simply determine which are the odd generators.

Let us now consider the matrix $\tilde{M}$

$$\tilde{M} = \begin{pmatrix} W_1 & \tilde{M}_2^1 & \tilde{M}_3^1 \\ \tilde{M}_1^2 & W_2 & \tilde{M}_3^2 \\ \tilde{M}_1^3 & \tilde{M}_2^3 & W_3 \end{pmatrix}. \tag{4.7}$$

The diagonal elements W_α ($\alpha = 1,2,3$) generate the Cartan subalgebra with weights w_α ($\alpha = 1,2,3$)

$$W_\alpha \Psi = w_\alpha \Psi. \tag{4.8}$$

In terms of the t-J model the weights are related to the z-component of the SU(2)-spin $S_z = 1/2(w_1 - w_2)$ and the number of electrons $Q = w_1 + w_2$. In order to calculate these weights for Bethe ansatz states we substitute (2.5) in (3.14), (3.15), (3.29), (3.30) and (3.31) and obtain with eq. (4.1) and (4.7) for $v \to \infty$

$$\begin{aligned} \left(1 - \frac{2}{v} W_1\right) \Psi + O(v^{-2}) &= \left(1 - \frac{2}{v}(L-N)\right) \Psi + O(v^{-2}), \\ \left(1 - \frac{2}{v} W_2\right) \Psi + O(v^{-2}) &= \left(1 - \frac{2}{v}(N-M)\right) \Psi + O(v^{-2}), \\ \left(-1 - \frac{2}{v} W_3\right) \Psi + O(v^{-2}) &= \left(-1 - \frac{2}{v} M\right) \Psi + O(v^{-2}). \end{aligned} \tag{4.9}$$

Therefore the weights can be expressed in terms of the quantities L (= number of sites), N (= number of first level roots) and M (= number of second level roots)

$$w_1 = n_\uparrow = L - N, \quad w_2 = n_\downarrow = N - M \quad w_3 = n_h = M \tag{4.10}$$

where $n_\uparrow$, $n_\downarrow$, n_h are the numbers of up-spins, down-spins and holes, respectively. At the end of this section we will derive inequalities between these weights and give a physical interpretation.

Next we show that the Bethe vectors are highest weight vectors with respect to the spl(2,1) superalgebra, i.e.,

$$\tilde{M}_\alpha^\beta \Psi = 0, \qquad \beta > \alpha. \tag{4.11}$$

For $\alpha = 1$, $\beta = 2$ or 3 we have, after using (3.8) and (4.2)

$$\begin{aligned} \tilde{M}_1^\beta \Psi = \sum_{j=1}^{N} & \sigma_{\beta(\alpha_1,\ldots,\alpha_{j-1})} B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \ldots B_{\alpha_{j-1}}(v_{j-1}) \\ & \times [\tilde{M}_1^\beta, B_{\alpha_j}(v_j)]_\pm B_{\alpha_{j+1}}(v_{j+1}) \ldots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}}, \end{aligned} \tag{4.12}$$

where

$$[\tilde{M}_1^\beta, B_\alpha(v)]_\pm = \tilde{M}_1^\beta B_\alpha(v) - \sigma_{\beta\alpha} B_\alpha(v)\tilde{M}_1^\beta = \delta_\alpha^\beta A(v) - \sigma_{\beta\alpha}\widetilde{T}_\alpha^\beta(v). \tag{4.13}$$

In order to commute $A(v_j)$ and $\widetilde{T}_\alpha^\beta(v_j)$ through the B_α's toward Φ we use the commutation rules (3.11), (3.12) and the property (2.13). Although many terms appear, it is possible to arrange them as follows:

$$\begin{aligned}\tilde{M}_1^\beta \Psi = \sum_{j=1}^{N} Y_{j\,\alpha_j}^{\beta}(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1}) B_{\alpha_{j+1}}(v_{j+1}) \dots \\ \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}},\end{aligned} \tag{4.14}$$

with yet unknown coefficients $Y_{j\,\alpha_j}^{\beta}$. The first coefficient, $Y_{1\,\alpha_1}^{\beta}$, can be obtained by using the first term in (3.11) and (3.12) when commuting $A(v_1)$ and $\widetilde{T}_\alpha^\beta(v_1)$ with $B_{\alpha_2}(v_2)B_{\alpha_3}(v_3)\dots B_{\alpha_N}(v_N)$, since otherwise the argument v_1 reappears in the B_α's. The contribution of the $A(v_1)$ term to $Y_{1\,\alpha_1}^{\beta}$ is straightforward, whereas for the $\widetilde{T}_\alpha^\beta(v_1)$ term we shall use the relation

$$-\frac{1}{2}\operatorname*{Res}_{v'=v} S_{\alpha\beta}^{\gamma\delta}(v - v') = \delta_\beta^\gamma \delta_\alpha^\delta \tag{4.15}$$

to get the eigenvalue problem for the transfer matrix $\tau_{(1)}(v_1, \{v_i\})$ (3.17). Once again, we just take the first term in (3.23) and (3.24) when commuting $A_{(1)}$ and $D_{(1)}$ with the $B_{(1)}$'s. Then, after some manipulations we have

$$Y_{1\,\alpha_1}^{\beta} = \delta_{\alpha_1}^{\beta}\left(a^L(v_1)\prod_{i=2}^{N}\frac{a(v_i - v_1)}{b(v_i - v_1)} - b^L(v_1)\prod_{i=2}^{N}\frac{a(v_1 - v_i)}{b(v_1 - v_i)}\prod_{\beta=1}^{M}\frac{a(\gamma_\beta - v_1)}{b(\gamma_\beta - v_1)}\right). \tag{4.16}$$

Analogous expressions follow for the other coefficients $Y_{j\,\alpha_j}^{\beta}$ $(j \geq 2)$

$$\begin{aligned}Y_{j\,\alpha_j}^{\beta} \propto \delta_{\alpha_j}^{\beta}\left(a^L(v_j)\prod_{i\neq j}^{N}\frac{a(v_i - v_j)}{b(v_i - v_j)} - b^L(v_j)\prod_{i\neq j}^{N}\frac{a(v_j - v_i)}{b(v_j - v_i)}\prod_{\beta=1}^{M}\frac{a(\gamma_\beta - v_j)}{b(\gamma_\beta - v_j)}\right), \\ j = 1, \dots, N.\end{aligned} \tag{4.17}$$

We observe that the requirement $Y_{j\,\alpha_j}^{\beta} = 0$ $(j = 1, \dots N,\ \beta = 2, 3)$ is equivalent to the Bethe ansatz equations (3.34), therefore Bethe states fulfil the highest weight condition $\tilde{M}_1^\beta \Psi = 0$ ($\beta = 2$ or 3).

To calculate $\tilde{M}_2^3\Psi$ we use the relation

$$\tilde{M}_2^3\Psi = \sigma_{3\{\alpha'\}}B_{\alpha_1'}(v_1)B_{\alpha_2'}(v_2)\dots B_{\alpha_{N'}}(v_N)\Phi\tilde{M}_{(1)2\{\alpha\}}^{3\{\alpha'\}}\Psi_{(1)}^{\{\alpha\}} \tag{4.18}$$

which follows from (4.2). $\tilde{M}_{(1)}$ is defined by asymptotic expansion of the monodromy $T_{(1)}$, in analogy with $\tilde{M}$ given by eq. (4.1). From (3.22) and commutation relations for $\tilde{M}_{(1)}$ and $\tilde{T}_{(1)}$ analogous to eq. (4.2) we get

$$\begin{aligned}\tilde{M}_{(1)2}{}^3\Psi_{(1)} = &\sum_{\beta=1}^{M}(-1)^{\beta-1}B_{(1)}(\gamma_1,\{v_i\})\dots B_{(1)}(\gamma_{\beta-1},\{v_i\}) \\ &\times[\tilde{M}_{(1)2}{}^3, B_{(1)}(\gamma_\beta,\{v_i\})]_+ B_{(1)}(\gamma_{\beta+1},\{v_i\})\dots B_{(1)}(\gamma_M,\{v_i\})\Phi_{(1)},\end{aligned} \tag{4.19}$$

where

$$[\tilde{M}_{(1)2}{}^3, B_{(1)}(\gamma)]_+ = A_{(1)}(\gamma) + \tilde{T}_{(1)3}{}^3(\gamma). \tag{4.20}$$

Analogously, by commuting $A_{(1)} + \tilde{T}_{(1)3}{}^3$ through the $B_{(1)}$'s we have

$$\begin{aligned}\tilde{M}_{(1)2}{}^3\Psi_{(1)} = &\sum_{\beta=1}^{M}Y_{(1),\beta}(\gamma_\beta,\{\gamma_\alpha\},\{v_i\})B_{(1)}(\gamma_1,\{v_i\})\dots \\ &\dots B_{(1)}(\gamma_{\beta-1},\{v_i\})B_{(1)}(\gamma_{\beta+1},\{v_i\})\dots B_{(1)}(\gamma_M,\{v_i\})\Phi_{(1)}\,.\end{aligned} \tag{4.21}$$

The coefficients $Y_{(1),\beta}$ can be derived in a straightforward way by taking the first terms of the commutation relations (3.23) and (3.24). We get

$$Y_{(1),\beta} = \prod_{i=1}^{N}a(\gamma_\beta - v_i)\prod_{\alpha\neq\beta}^{M}\frac{a(\gamma_\alpha-\gamma_\beta)}{b(\gamma_\alpha-\gamma_\beta)} + (-1)^M\prod_{i=1}^{N}b(\gamma_\beta - v_i)\prod_{\alpha\neq\beta}^{M}\frac{w(\gamma_\beta-\gamma_\alpha)}{b(\gamma_\beta-\gamma_\alpha)}, \tag{4.22}$$
$$\beta = 1,\dots,M.$$

The requirement $Y_{(1),\beta} = 0$ $(\beta = 1,\dots,M)$ is equivalent to the Bethe ansatz equations (3.35), which implies $\tilde{M}_2^3\Psi = 0$. We stress that the property (4.11) can also be proved for the other two choices of pseudovacuum in a similar way.

At the end of this section we derive some inequalities between the weights w_α $(\alpha = 1,2,3)$. From eq.(4.5) we have

$$[\tilde{M}_\alpha^\beta, \tilde{M}_\beta^\alpha]_\pm = W_\alpha^\alpha \pm W_\beta^\beta\,, \quad \beta > \alpha. \tag{4.23}$$

Using $(\tilde{M}_\alpha^\beta)^\dagger = \tilde{M}_\beta^\alpha$ and the highest weight property of the Bethe vectors (4.11) we obtain

$$w_1 \geq w_2 \geq -w_3. \tag{4.24}$$

Combining (4.10) with $w_i \geq 0$ $(i = 1, 2, 3)$ and (4.24) we find conditions for the numbers N and M of roots in the first and second level Bethe ansatz, respectively

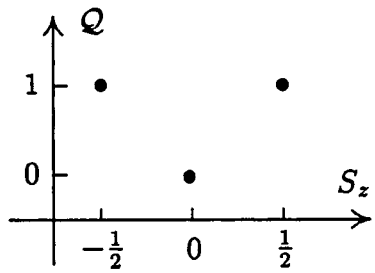
$$M \leq N \leq \frac{L+M}{2}\,, \quad 0 \leq M \leq L. \tag{4.25}$$

This means in terms of physical quantities that the magnetization $S_z = \frac{1}{2}(n_\uparrow - n_\downarrow) = \frac{1}{2}(L - 2N + M)$ and the number of electrons $\mathcal{Q} = n_\uparrow + n_\downarrow = L - M$ are restricted to $0 \leq S_z \leq \mathcal{Q}/2 \leq L/2$.

5. RESULTS FOR SMALL AND LARGE LATTICES

In this section we illustrate the algebraic properties of the Bethe states. We begin with a lattice of two sites and then discuss the case of lattices with a large number of sites.

The simple case of one lattice point corresponds to the fundamental representation of spl(2,1) which is given by the following weight diagram in the $(S_z, \mathcal{Q})$-plane, where $\mathcal{Q}$ is the number of electrons and S_z the total magnetization of the system.

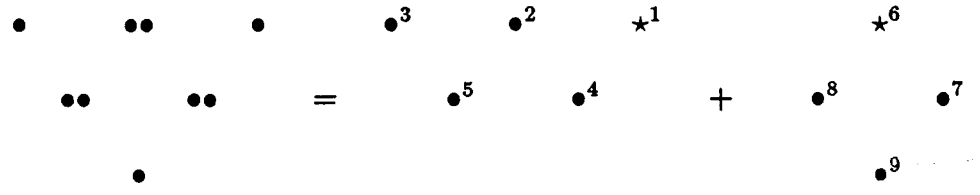


By diagonalization of the t-J hamiltonian (2.17) (or of the transfer matrix τ) on a lattice with two sites we obtain the eigenstates and eigenvalues

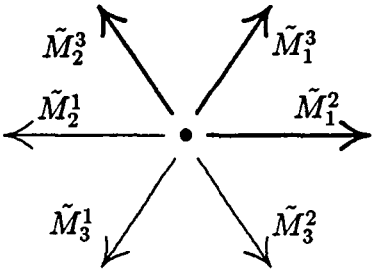
$$\begin{aligned}
\Psi_1 &= |\uparrow\uparrow\rangle\,, && E = 2\\
\Psi_2 &= \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\,, && E = 2\\
\Psi_3 &= |\downarrow\downarrow\rangle\,, && E = 2\\
\Psi_4 &= \tfrac{1}{\sqrt{2}}(|0\ \uparrow\rangle + |\uparrow\ 0\rangle)\,, && E = 2\\
\Psi_5 &= \tfrac{1}{\sqrt{2}}(|0\ \downarrow\rangle + |\downarrow\ 0\rangle)\,, && E = 2
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
\Psi_6 &= \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\,, \qquad E=-2 \\
\Psi_7 &= \tfrac{1}{\sqrt{2}}(|0\uparrow\rangle - |\uparrow 0\rangle)\,, \qquad E=-2 \\
\Psi_8 &= \tfrac{1}{\sqrt{2}}(|0\downarrow\rangle - |\downarrow 0\rangle)\,, \qquad E=-2 \\
\Psi_9 &= |0\,0\rangle\,, \qquad E=-2
\end{aligned} \tag{5.1'}$$

where 0 denotes a hole. This result can be visualized in terms of the following spl(2,1) weight diagrams in the Clebsh-Gordan series $3 \otimes 3 = 5 \oplus 4$



The numbers in the weight diagrams specify the eigenvectors according to eqs. (5.1) and (5.1'). The symbol $\star$ denotes the highest weight vectors according to eq. (5.2) below. Notice that the ground state is degenerate and given by a quartet. All states of an irreducible representation can be generated by repeated application of the shift operators $\tilde{M}^{\beta}_{\alpha}$ $(\beta \neq \alpha)$ to any one of the states. Graphically, the effect of the shift operators on a general state of a representation of spl(2,1) is given by



On the other hand, if we solve the Bethe ansatz equations (3.36) and (3.37) for two sites we obtain only two eigenvectors, $\Psi_1 = \Phi$ and $\Psi_6 = B_{\alpha_1}(v_1 = 0)\Phi\,\Phi^{\alpha_1}_{(1)}$, with energy eigenvalues 2 and -2, respectively (see eq. (3.49)). In the language of the nested Bethe ansatz Φ and $\Phi_{(1)}$ are the first and second level pseudoground states, respectively. We can easily check that these eigenvectors are highest weight vectors of the spl(2,1) superalgebra, in agreement with our general proof in section 4.

$$\tilde{M}^{\beta}_{\alpha}\Psi_1 = \tilde{M}^{\beta}_{\alpha}\Psi_6 = 0\,, \quad \beta > \alpha\,. \tag{5.2}$$

Furthermore, the seven missing eigenvectors can be obtained by successive applications of the shift operators

$$\begin{aligned} \Psi_2 &= \tilde{M}_2^1 \Psi_1 & \qquad \Psi_7 &= \tilde{M}_3^2 \Psi_6 \\ \Psi_3 &= (\tilde{M}_2^1)^2 \Psi_1 & \qquad \Psi_8 &= \tilde{M}_3^1 \Psi_6 \\ \Psi_4 &= \tilde{M}_3^1 \Psi_1 & \qquad \Psi_9 &= \tilde{M}_3^1 \tilde{M}_3^2 \Psi_6 \ . \\ \Psi_5 &= \tilde{M}_2^1 \tilde{M}_3^1 \Psi_1 \end{aligned} \tag{5.3}$$

Therefore, the Bethe ansatz together with the supersymmetry of the model provide all 9 eigenvectors for the 2-sites model.

We remark that by solving all three different forms of the BAE we get all highest weight vectors of the SU(2) algebra. Solving eqs. (3.38) and (3.39) we get the eigenvectors Ψ_4 and Ψ_9 and from eqs.(3.40) and (3.41) we obtain the eigenvectors Ψ_1 and Ψ_7.

In the case of lattices with a large number of sites the Bethe ansatz method turns out to be crucial, since the effort of an exact diagonalization growths exponentially with the number of sites L. As already pointed out in sections 3 and 4, by this method, the problem of finding the spectrum of the t-J hamiltonian reduces to the solution of the BAE's (3.36) and (3.37) for the parameters v's and γ's. Adopting the string conjecture, which has an accuracy of $O(e^{-L})$, the solutions of the BAE's are parametrized in terms of the numbers I_α^n and J_β. Moreover, each set $\{I_\alpha^n, J_\beta\}$ where the I's and J's are pairwise different specify a Bethe vector, which is the leading vector of an spl(2,1) multiplet.

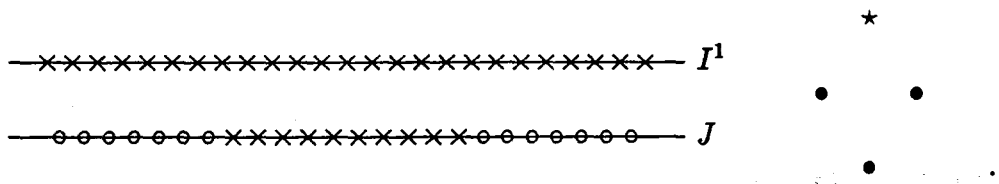
Now we illustrate our results for the ground state and some elementary excitations at "half-filling" $F = \mathcal{Q}/L = 1$. The following holds true for any lattice size, especially in the thermodynamic limit $L \to \infty$.

5.1) GROUND STATE

The ground state involves only real roots. This can be proved as usual by minimizing the free energy for finite temperature T and taking $T \to 0$. A complete analysis of this proof as well as a detailed investigation of the elementary excitations of this model is given in the next chapter. The ground state, as we already remarked, corresponds to the configuration $N_1 = N = \frac{L}{2}$, $N_{2,3,...} = 0$ and $M = 0$. In this configuration the number of vacancies for the first-level and second-level real roots is $2I_{max}^1 + 1 = \frac{L}{2}$ and

The quantum numbers of this state are $S_z = 0$ and $\mathcal{Q} = 40 = L - 1$, which means a holon is a particle-like excitation with spin 0 and charge -1.

At arbitrary filling $F < 1$, for the ground state the distribution of the roots in the v-plane also involves only real roots. In contrast to the half-filling case there is now in addition a "sea" of real roots in the γ-plane, such that there appears a nontrivial Fermi-level. For example, for a lattice of $L = 40$ sites we find a Bethe ansatz state with $N = 25$ first level real roots and $M = 10$ second level real roots and 14 BA-holes



Also here the ground state is member of a quartet. The quantum numbers are $S_z = 0$ and $\mathcal{Q} = 30$, which means spin 0 and filling $F = 1 - 10/40 = .75$. Due to the nontrivial Fermi-level there exist "holon-antiholon" excitations in this case. A more quantitative description of the excitation spectrum can be found in section 4 of the next chapter.

6. COMPLETENESS OF THE BETHE VECTORS

In this section we show how to construct a complete set of eigenvectors of the t-J hamiltonian for arbitrary chain of length L. This is obtained by combining the Bethe ansatz with the supersymmetry of the model.

From the section 3 we know that all collections $\{I^n_\alpha, J_\beta\}$ where the I's and J's are pairwise different specify all the Bethe vectors $(|\psi_{Bethe}\rangle_{N,M})$. The number of admissible values for the I^n_α's and the J_β's (for fixed $\{N_n\}$ and M) is $(2I^n_{\max} + 1)$ and $(2J_{\max} + 1)$, respectively. $I^n_{\max}$ and $J_{\max}$ are given by eqs. (3.47) and (3.48). Taking into account that many different string configurations N_n give the same number of roots N (see eq. (3.43)), the number of possible Bethe vectors for fixed N, M is given by

$$Z(N,M) = \sum_{\{N_n\}} \binom{2J_{\max}+1}{M} \prod_n \binom{2I^n_{\max}+1}{N_n}, \tag{6.1}$$

where the sum over $\{N_n\}$ is constrained to $\sum_n nN_n = N$. It is convenient to introduce the quantity $q = \sum_n N_n$. Using eqs. (3.47) and (3.48) we write this sum as

$$Z(N,M) = \sum_{q=0}^{N} \binom{q-1}{M} \sum_{\{N_n\}} \prod_n \binom{L - \sum_m t_{nm}N_m + M}{N_n}, \tag{6.2}$$

where the inner sum is constrained to fixed values of N and q. This expression resembles the one calculated by Bethe in the isotropic Heisenberg model [15,23] and can be simplified to

$$Z(N,M)=\sum_{q=0}^{N}\frac{L+M-2N+1}{L+M-N+1}\binom{q-1}{M}\binom{L+M-N+1}{q}\binom{N-1}{q-1}. \tag{6.3}$$

The total number of Bethe vectors is obtained by summing $Z(N,M)$ over all N,M restricted to (4.25). However, this number is less than 3^L, so that the Bethe ansatz does not yield all the states of the model. In order to construct a complete set we shall invoke the supersymmetry of the transfer matrix. First, from (4.6) it follows that the Bethe vectors are classified by multiplets corresponding to irreducible representations of the superalgebra spl(2,1). Furthermore, from (4.11) follows that the Bethe vectors are highest weight vectors. Then by acting with the spl(2,1) lowering operators $\tilde{M}_\alpha^\beta$ $(\beta<\alpha)$ on the Bethe states we obtain additional states Each Bethe state (with fixed N,M in the interval (4.25)) is the highest weight vector in a multiplet of dimension [43]

$$d(N,M)=\begin{cases}4S_z+1=2L+1 & \text{if } N=M=0\\ 8(S_z+1/2)=4(L-2N+M+1) & \text{otherwise.}\end{cases} \tag{6.4}$$

With these considerations, the total number of eigenvectors is

$$\begin{aligned}Z&=\sum_{M=0}^{L}\sum_{N=M}^{\frac{L+M}{2}}d(N,M)Z(N,M)=2L+1+Z_1-1+Z_2\\ &=2L+1+4\sum_{N=1}^{\frac{L}{2}}(L-2N+1)\frac{L-2N+1}{L-N+1}\sum_{q=1}^{N}\binom{L-N+1}{q}\binom{N-1}{q-1}\\ &\quad+4\sum_{M=1}^{L}\sum_{N=M}^{\frac{L+M}{2}}(L-2N+M+1)\frac{L+M-2N+1}{L+M-N+1}\\ &\quad\times\sum_{q=1}^{N}\binom{q-1}{M}\binom{L+M-N+1}{q}\binom{N-1}{q-1}.\end{aligned} \tag{6.5}$$

The first sum in (6.5) can be performed (see ref. [23]) to give

$$Z_1=4\cdot 2^L-4(L+1). \tag{6.6}$$

The second sum Z_2 deserves special attention. We present the main necessary steps for its evaluation. First, performing the sum over q we get

$$Z_2=4\sum_{M=1}^{L}\sum_{N=M}^{\frac{L+M}{2}}(L-2N+M+1)\frac{L+M-2N+1}{L+M-N+1}\binom{N-1}{M}\binom{L}{N}. \tag{6.7}$$

Employing some combinatorics and making the substitution $N \to x = N - M$ we obtain

$$Z_2 = 4 \sum_{M=1}^{L} \sum_{x=0}^{\frac{L-M}{2}} (L-2x-M+1) \left[\binom{L}{x+M} \binom{x+M-1}{M} - \binom{L}{x-1} \binom{L-x}{M} \right]. \quad (6.8)$$

After some rearrangements this expression can be rewritten as

$$\begin{aligned} Z_2 = 4 \sum_{M=1}^{L} \sum_{x=0}^{\frac{L-M}{2}} L \Bigg[& \left[\binom{L-1}{x+M} \binom{x+M-1}{M} + \binom{L-1}{x-2} \binom{L-x}{M} \right] \\ & - (M+1) \left[\binom{L}{x+M} \binom{x+M-1}{x-2} + \binom{L}{x-1} \binom{L-x}{M+1} \right] \Bigg]. \end{aligned} \quad (6.9)$$

Substituting $x \to L - x - M + 1$ in the second and fourth terms of eq. (6.9) we get

$$Z_2 = 4 \sum_{M=1}^{L} \sum_{x=0}^{L-M+1} \left[L \binom{L-1}{x+M} \binom{x+M-1}{M} - (M+1) \left[\binom{L}{x+M} \binom{x+M-1}{M+1} \right] \right]. \quad (6.10)$$

Using the binomial formula we obtain after some rearrangements

$$Z_2 = 4L! \sum_{M=1}^{L} \frac{1}{M!(L-2-M)!} \int_0^1 p^M \left[(p+1)^{L-M-2} (1-p) \right] dp. \quad (6.11)$$

Interchanging the sum and the integral and performing the sum gives

$$Z_2 = \frac{4L!}{(L-2)!} \int_0^1 (1-p) \left[(1+2p)^{L-2} - (1+p)^{L-2} \right] dp. \quad (6.12)$$

This integral can be easily performed, resulting in

$$Z_2 = 3^L - 4 \cdot 2^L + 2L + 3. \quad (6.13)$$

Substituting (6.6) and (6.13) into (6.5) we get

$$Z = 3^L. \quad (6.14)$$

Thus we have shown that the number of eigenvectors of the t-J hamiltonian is 3^L, which is precisely the number of states in the Hilbert space of a chain of length L, where at each site there may be either a spin up or a spin down electron or a hole.

7. SUMMARY

In this chapter we have shown that the Bethe ansatz states for the one-dimensional supersymmetric t-J model are highest weight vectors of an spl(2,1) superalgebra. Then, by acting with the spl(2,1) lowering operators on the Bethe states we have obtained a complete set of eigenvectors of the t-J hamiltonian.

An interesting extension of this work is an analysis of the $spl_q(2,1)$ "quantum-group" structure of a "q-deformed" version of this model. This is presented in chapter 4.

APPENDIX A : The cancellation of the unwanted terms and the BAE

In this appendix we show that the cancellation conditions of the "unwanted terms" $u.t. = u.t.(A) + u.t.(D)$ and $u.t._{(1)} = u.t.(A_{(1)}) + u.t.(D_{(1)})$ are equivalent to the Bethe ansatz equations (3.34) and (3.35). As already pointed out in section 3 all terms whose arguments are exchanged when $A(v)$ and $\tau_D(v)$ is commuted through $\prod_{j=1}^{N} B_{\alpha_j}(v_j)$ using eqs.(3.11) and (3.12) are called $u.t.(A)$ and $u.t.(D)$, respectively. They can be arranged as follows

$$u.t.(A) = \sum_{j=1}^{N} K_j^{(A)}(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1}) B_{\alpha_j}(v) \times B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}} , \tag{A.1}$$

$$u.t.(D) = \sum_{j=1}^{N} K_j^{(D)}(v_j, \{v_i\}) B_{\alpha_1}(v_1) B_{\alpha_2}(v_2) \dots B_{\alpha_{j-1}}(v_{j-1}) B_{\alpha_j}(v) \times B_{\alpha_{j+1}}(v_{j+1}) \dots B_{\alpha_N}(v_N) \Phi \Psi_{(1)}^{\{\alpha\}} . \tag{A.2}$$

Here $K_j^{(A)}$ and $K_j^{(D)}$ $(j = 1, \dots, N)$ are coefficients to be determined. The first coefficient of eq.(A.1) can be computed using the second term in (3.11) when commuting $A(v)$ with $B_{\alpha_1}(v_1)$ and then using the first term in eq.(3.11) when commuting $A(v_1)$ with the remaining B_α's, since otherwise the argument v_1 reappears in the B_α's . We get

$$K_1^{(A)} = -a^L(v_1) \frac{c(v_1 - v)}{b(v_1 - v)} \prod_{i \neq 1}^{N} \frac{a(v_i - v_1)}{b(v_i - v_1)}. \tag{A.3}$$

In order to calculate $K_j^{(D)}$ we rewrite the second term of eq.(3.12) as

$$-\frac{1}{v - v'} \operatorname*{Res}_{v'' = v'} \left(S_{\gamma' \alpha}^{\gamma'' \alpha'}(v'' - v') B_{\alpha'}(v) T_{\gamma''}^{\gamma}(v') \right), \tag{A.4}$$

by means of eqs.(2.5) and (4.15). Then, proceeding along the same lines as in the calculation of $K_1^{(A)}$ we get the eigenvalue problem for the transfer matrix $\tau_{(1)}$ (3.17). In addition, just taking the first term in eqs.(3.23) and (3.24) when passing $A_{(1)}$ and $\widetilde{T}_3^3$ through the $B_{(1)}$'s we obtain, after some rearrangements

$$K_1^{(D)} = -b^L(v_1) \frac{c(v - v_1)}{b(v - v_1)} \prod_{i \neq 1}^{N} \frac{a(v_1 - v_i)}{b(v_1 - v_i)} \prod_{\beta = 1}^{M} \frac{a(\gamma_\beta - v_1)}{b(\gamma_\beta - v_1)} \tag{A.5}$$

To get the other coefficients $K_j^{(A)}$ and $K_j^{(D)}$ $(j = 2, \ldots, N)$ we use the commutation rules for the B_α's (3.13) and put $B_{\alpha_j}(v)$ in the first place . Then, repeating the same procedure we obtain analogous expressions with j in the place of 1. Furthermore, the requirement $K_j^{(A)} + K_j^{(D)} = 0$ $(j = 1, \ldots, N)$ together with the fact that $\frac{c(v)}{b(v)}$ is an odd function (see eq.(2.5)) leads to the Bethe ansatz equation (3.34).

The "unwanted terms" that appear in the second level of the Bethe ansatz method can be arranged as follows

$$\begin{aligned} u.t._{(1)} = \sum_{\beta=1}^{M} \left(K_\beta^{(A_{(1)})} + K_\beta^{(D_{(1)})} \right) B_{(1)}(\gamma_1, \{v_i\}) B_{(1)}(\gamma_2, \{v_i\}) \ldots \\ \ldots B_{(1)}(\gamma_{\beta-1}, \{v_i\}) B_{(1)}(v, \{v_i\}) B_{(1)}(\gamma_{\beta+1}, \{v_i\}) \ldots B_{(1)}(\gamma_M, \{v_i\}) \Phi_{(1)} \, . \end{aligned} \tag{A.6}$$

By similar arguments as above, the coefficients $K_1^{(A_{(1)})} + K_1^{(D_{(1)})}$ can be computed using the second term in eqs.(3.23) and (3.24) when commuting $A_{(1)}(v, \{v_i\})$ and $\widetilde{T}_{(1)3}{}^{3}(v, \{v_i\})$ through $B_{(1)}(\gamma_1, \{v_i\})$ and then using the first term in (3.23) and (3.24) when commuting $A_{(1)}(\gamma_1, \{v_i\})$ and $\widetilde{T}_{(1)3}{}^{3}(\gamma_1, \{v_i\})$ with the remaining $B_{(1)}$'s

$$\begin{aligned} K_1^{(A_{(1)})} + K_1^{(D_{(1)})} = & - \prod_{i=1}^{N} a(\gamma_1 - v_i) \frac{c(\gamma_1 - v)}{b(\gamma_1 - v)} \prod_{\alpha \neq 1}^{M} \frac{a(\gamma_\alpha - \gamma_1)}{b(\gamma_\alpha - \gamma_1)} \\ & - (-1)^M \frac{c(v - \gamma_1)}{b(v - \gamma_1)} \prod_{i=1}^{N} b(\gamma_1 - v_i) \prod_{\alpha \neq 1}^{M} \frac{w(\gamma_1 - \gamma_\alpha)}{b(\gamma_1 - \gamma_\alpha)}. \end{aligned} \tag{A.7}$$

Once again, the other coefficients can be obtained using the commutation rules (3.25). The requirement $K_\beta^{(A_{(1)})} + K_\beta^{(D_{(1)})} = 0$ $(\beta = 1, \ldots, M)$ together with the fact that $\frac{c(v)}{b(v)}$ is an odd function yields the Bethe ansatz equation (3.35).

Chapter 3

Thermodynamic Properties of the Supersymmetric t-J Model

1. INTRODUCTION

In this chapter we present a detailed analysis of the ground state and elementary excitation spectrum of the supersymmetric t-J model in one dimension. In the thermodynamic limit ($L \to \infty$) we prove by minimizing the free energy that the ground state at zero temperature involves only real roots. We interpret the spectra in terms of soliton-like excitations which are identified as holons and spinons. The dispersion relations for some physical situations are also derived.

For a system at arbitrary filling in the presence of an external magnetic field non-trivial Fermi-levels appear. We compute them analytically for some special situations (for example, system near half-filling) as well as numerically. In this case exists an excitation involving the transfer of a root below the Fermi level to a previously unoccupied state above the Fermi level. This excitation, which does not change the number of particles is referred as holon-antiholon excitation. The relation between physical quantities as the filling and the chemical potential is also derived. The methods used here are based on the works of Faddeev et al. [23,50] and Takahashi [47] for the isotropic Heisenberg model and Lieb [51], Yang [22] and Korepin [52] for the one-dimensional Bose-gas system with repulsive delta function potential. Part of the results obtained in this chapter were derived by Bares and Blater [40] using another form of the Bethe-ansatz equations of the previous chapter (3.38) – (3.39). This approach leads to involved manipulations

of the ground state (formed by two-strings in the v-axis) and the excitation spectrum. Of course, the results for the physical quantities are the same.

This chapter is arranged as follows. In section 2 we find the structure of the ground state of the model. The Yang's equations are also obtained. In section 3 we derive the relation between some physical quantities of interest, as the magnetization and magnetic field. We also introduce the "Fermi levels" of the system and their dependence on the external parameters of the model. A quantitative description of the low-lying excitation spectrum of the system as well as the corresponding dispersion laws can be found in section 4. Section 5 summarizes our main results.

2. THE THERMODYNAMIC EQUATIONS AND GROUND STATE

In this section we describe the thermodynamic properties of the one-dimensional supersymmetric t-J model and find the ground state specially at zero temperature.

In the limit $L \to \infty$ the roots v and γ of the BAE tend to have a continuous distribution with densities

$$\rho_n(v) \equiv \frac{1}{L}\sum_{\alpha=1}^{N_n}\delta(v-v_\alpha^n), \tag{2.1}$$

$$\sigma(\gamma) \equiv \frac{1}{L}\sum_{\xi=1}^{M}\delta(\gamma-\gamma_\xi). \tag{2.2}$$

We also define the densities of BA-holes $\rho_n^h(v)$, $\sigma^h(\gamma)$ as

$$\frac{dg_n(v)}{dv} = 2\pi\left(\rho_n(v)+\rho_n^h(v)\right), \tag{2.3}$$

$$\frac{dh(\gamma)}{d\gamma} = 2\pi\left(\sigma(\gamma)+\sigma^h(\gamma)\right), \tag{2.4}$$

where $g_n(v)$ and $h(\gamma)$ are defined by the expressions

$$g_n(v) \equiv \theta\left(\frac{v_\alpha^n}{n}\right) - \frac{1}{L}\sum_m\sum_{\beta=1}^{N_m}\Theta_{nm}(v-v_\beta^m) + \frac{1}{L}\sum_{\beta=1}^{M}\theta\left(\frac{v_\alpha^n-\gamma_\beta}{n}\right), \tag{2.5}$$

$$h(\gamma) \equiv \frac{1}{L}\sum_n\sum_{\alpha=1}^{N_n}\theta\left(\frac{v_\alpha^n-\gamma_\beta}{n}\right), \tag{2.6}$$

appearing in the BAE (eqs.(3.44) and (3.45) of the previous chapter). The definition of the densities of BA-holes by (2.3)-(2.4) are motivated by the fact that e.g. the BAE (3.44) imply for all $I^{(n)}_{i(j)} \leftrightarrow v^{(n)}_{i(j)}$

$$\begin{aligned}\int_{v_i^{(n)}}^{v_j^{(n)}} dv \left(\frac{dg_n(v)}{dv} - 2\pi\rho_n(v)\right) &= I_j^{(n)} - I_i^{(n)} \; - \; \#roots \in [I_i^{(n)}, I_j^{(n)}] \\ &= \#vacancies \; - \; \#roots \\ &= \#BA - holes.\end{aligned}$$

Differentiating eqs (2.5), (2.6) and taking (2.3) and (2.4) into account we get

$$\frac{1}{\pi}\frac{n}{n^2+v^2} - \sum_{m\neq n} A_{nm}\rho_m(v) - B_{nn}\rho_n(v) + [n]\sigma(v) = \rho_n(v) + \rho_n^h(v), \tag{2.7}$$

$$\sum_n [n]\rho_n(\gamma) = \sigma(\gamma) + \sigma^h(\gamma), \tag{2.8}$$

where A_{nm} and B_{nn} are operators defined by

$$A_{nm} \equiv [|n-m|] + 2[|n-m|+2] + 2[|n-m|+4] + \ldots + 2[n+m-2] + [n+m],$$

$$B_{nn} \equiv 2[2] + 2[4] + \ldots + 2[2n-2] + [2n], \tag{2.9}$$

$[n]$ is an operator defined by (f is an arbitrary function)

$$\begin{aligned}[n]f(v) &\equiv \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{n}{n^2+(v-k)^2} f(k) \\ &= \frac{1}{\pi}\frac{n}{n^2+v^2} * f(v)\end{aligned} \tag{2.10}$$

and

$$[0]f(v) \equiv f(v).$$

Here we adopt the notation of ref. [47]. From eq. (2.9) we see that $A_{nn} = [0] + B_{nn}$. Then substituting this property into (2.7) we obtain

$$\frac{1}{\pi}\frac{n}{n^2+v^2} - \sum A_{nm}\rho_m(v) + [n]\sigma(v) = \rho_n^h(v) \tag{2.11}$$

and

$$\sum_n [n]\rho_n(\gamma) = \sigma(\gamma) + \sigma^h(\gamma). \tag{2.12}$$

These equations are solved in section 4 for some special cases.

In order to find the equilibrium state of the system at fixed temperature T, external magnetic filed H (≥ 0) and chemical potential A (≥ 0) we must minimize the thermodynamic potential (gran-potential) $\Omega = E - TS - AQ - HS_z$. Here E is the energy (eq. (3.50) of chapter 2) and S_z the magnetization (eq. (3.52) of chapter 2) of the system per site which can be written in terms of the densities of roots ρ_n, σ as

$$\frac{E}{L} = 1 - \sum_n \int_{-\infty}^{\infty} dv \frac{4n}{n^2+v^2} \rho_n(v), \tag{2.13}$$

$$\frac{S_z}{L} = \frac{1}{2} - \sum_n n \int_{-\infty}^{\infty} dv \rho_n(v) + \frac{1}{2} \int_{-\infty}^{\infty} d\gamma \sigma(\gamma). \tag{2.14}$$

For later use we also write the filling or the number of electrons Q (eq. (3.53) of chapter 2) in terms of σ as

$$F = \frac{Q}{L} = 1 - \int_{-\infty}^{\infty} d\gamma \sigma(\gamma). \tag{2.15}$$

We still need to find the entropy S of the system. It is given by the logarithm of the number of accessible states of the system. We follow Yang and use the fact that in a small interval $[v, v+dv]$ ($[\gamma, \gamma+d\gamma]$) there are $\rho_n(v)Ldv$ ($\sigma(\gamma)Ld\gamma$) particles or BA-roots and $\rho_n^h(v)Ldv$ ($\sigma^h(\gamma)Ld\gamma$) BA-holes. Therefore, assuming that these numbers are much larger compared to the unit, the total entropy in this small interval is

$$\begin{aligned} \ln \frac{[(\rho_n + \rho_n^h)Ldv]!}{(\rho_n Ldv)!(\rho_n^h Ldv)!} + \ln \frac{[(\sigma + \sigma^h)Ldv]!}{(\sigma Ldv)!(\sigma^h Ldv)!} = \\ = \left((\rho_n + \rho_n^h)\ln(\rho_n + \rho_n^h) - \rho_n \ln \rho_n - \rho_n^h \ln \rho_n^h\right) Ldv \cdot \\ + \left((\sigma + \sigma^h)\ln(\sigma + \sigma^h) - \sigma \ln \sigma - \sigma^h \ln \sigma^h\right) Ld\gamma \end{aligned}$$

Therefore the total entropy per site is given by

$$\begin{aligned} \frac{S}{L} = \sum_n \int_{-\infty}^{\infty} dv \Big((\rho_n + \rho_n^h)\ln(\rho_n + \rho_n^h) - \rho_n \ln \rho_n - \rho_n^h \ln \rho_n^h\Big) \\ + \int_{-\infty}^{\infty} d\gamma \Big((\sigma + \sigma^h)\ln(\sigma + \sigma^h) - \sigma \ln \sigma - \sigma^h \ln \sigma^h\Big) \end{aligned} \tag{2.16}$$

Now we can proceed on the minimization of the thermodynamic potential Ω

$$\begin{aligned} 0 = \frac{1}{L}\delta\Omega = \sum_n \int_{-\infty}^{\infty} dv \left(\frac{-4n}{n^2+v^2} + nH\right)\delta\rho_n - \int_{-\infty}^{\infty} d\gamma \left(\frac{H}{2} - A\right)\delta\sigma \\ -T\sum_n \int_{-\infty}^{\infty} dv \left(\delta\rho_n \ln\left(1+\frac{\rho_n^h}{\rho_n}\right)\delta\rho_n^h \ln\left(1+\frac{\rho_n}{\rho_n^h}\right)\right) \\ -T\int_{-\infty}^{\infty} d\gamma \left(\delta\sigma \ln\left(1+\frac{\sigma^h}{\sigma}\right) + \delta\sigma^h \ln\left(1+\frac{\sigma}{\sigma^h}\right)\right) \end{aligned} \quad . \tag{2.17}$$

From eqs.(2.11), (2.12) we can express $\delta\rho_n^h$ and $\delta\sigma^h$ in terms of $\delta\rho_n$ and $\delta\sigma$

$$\delta\rho_n^h = -\sum_m A_{nm}\delta\rho_m + [n]\delta\sigma, \tag{2.18}$$

$$\delta\sigma^h = -\sum_n [n]\rho_n - \delta\sigma.$$

Then substituting the expressions above into (2.17) and using the symmetry $A_{nm} = A_{mn}$, (see eq.(2.9)) we have

$$\begin{aligned} \sum_n \int_{-\infty}^{\infty} dv \Big\{ -\frac{4n}{n^2+v^2} + nH - T\ln\left(1+\frac{\rho_n^h}{\rho_n}\right) + T\sum_m A_{nm}\ln\left(1+\frac{\rho_m}{\rho_m^h}\right) \\ -T[n]\ln\left(1+\frac{\sigma}{\sigma^h}\right)\Big\}\delta\rho_n + \int_{-\infty}^{\infty} d\gamma \Big\{ -\frac{H}{2} + A - T\sum_n [n]\ln\left(1+\frac{\rho_m}{\rho_m^h}\right) \\ -T\ln\left(1+\frac{\sigma^h}{\sigma}\right) + T\ln\left(1+\frac{\sigma}{\sigma^h}\right)\Big\}\delta\sigma = 0 \end{aligned} \quad . \tag{2.19}$$

Since the variations $\delta\rho_n(v)$ and $\delta\sigma(\gamma)$ are independent, we obtain directly from this equation

$$\ln\left(1+\frac{\rho_n^h}{\rho_n}\right) = \frac{-\frac{4n}{n^2+v^2} + nH}{T} + \sum_m A_{nm}\ln\left(1+\frac{\rho_m}{\rho_m^h}\right) - [n]\ln\left(1+\frac{\sigma}{\sigma^h}\right), \tag{2.20}$$

$$\ln\left(1+\frac{\sigma^h}{\sigma}\right) = \frac{-\frac{H}{2}+A}{T} - T\sum_n [n]\ln\left(1+\frac{\rho_n}{\rho_n^h}\right) + \ln\left(1+\frac{\sigma}{\sigma^h}\right).$$

Using the properties

$$[n]const. = const., \quad [n][m] = [n+m],$$

$$[j]A_{nm} = \begin{cases} A_{n,m+j}, & m \geq n \\ A_{n+j,m}, & m < n \end{cases}, \tag{2.21}$$

and

$$([0]+[2])\ln\left(1+\frac{\rho_{n+1}^h}{\rho_{n+1}}\right)=[1]\left\{\ln\left(1+\frac{\rho_n^h}{\rho_n}\right)+\ln\left(1+\frac{\rho_{n+2}^h}{\rho_{n+2}}\right)\right\}.$$

We derive the following relations, after a cumbersome calculation (see Appendix A)

$$\begin{aligned}K_1(v)=&-\frac{4}{1+v^2}+H+T[2]\ln\left(1+e^{-\frac{K_1(v)}{T}}\right)-T[1]\ln\left(1+e^{-\frac{\epsilon(v)}{T}}\right)\\&+T([0]+[2])\sum_j[j]\ln\left(1+e^{-\frac{K_{j+1}(v)}{T}}\right)\end{aligned}\tag{2.22}$$

$$\begin{aligned}K_n(v)=&H+T[1]\ln\left(1+e^{-\frac{K_{n-1}(v)}{T}}\right)+T[2]\ln\left(1+e^{-\frac{K_n(v)}{T}}\right)\\&+T([0]+[2])\sum_j[j]\ln\left(1+e^{-\frac{K_{j+n}(v)}{T}}\right);\quad n\geq 2\end{aligned}\tag{2.23}$$

$$\epsilon(\gamma)=-\frac{H}{2}+A-T\sum_n[n]\ln\left(1+e^{-\frac{K_n(\gamma)}{T}}\right),\tag{2.24}$$

where K_n and ϵ are defined by

$$K_n(v)\equiv T\ln\frac{\rho_n^h(v)}{\rho_n(v)},\tag{2.25}$$

$$\epsilon(\gamma)\equiv T\ln\frac{\sigma^h(\gamma)}{\sigma_n(\gamma)}.\tag{2.26}$$

The coupled nonlinear integral equations for the functions $K_1(v)$ (2.22), $K_n(v)$ (2.23) and $\epsilon(\gamma)$ (2.24) play a central role in the investigation of the thermodynamic properties of the model. In fact, this kind of equations was introduced by Yang and Yang [22] for the one-dimensional Bose gas with delta-function interaction. In this case the situation is simpler, since just one integral equation appears. Equations of this type are commonly called Yang equations. They were also derived for the isotropic [47] and anisotropic [53] Heisenberg model, Hubbard model [54], Kondo model [55], non degenerate Anderson model [56], isotropic [57] and anisotropic Heisenberg model [58] with arbitrary spin. The physical interpretation of the functions $K_1(v)$ and $\epsilon(\gamma)$ is that they provide directly the elementary excitation spectrum. This can be motivated by considering the ratio $f(v)$ of the number of occupied vacancies to the number of all possible vacancies in the interval v, $v+dv$, which according to eqs.(2.25) reads

$$f(v)=\frac{\rho_1(v)}{\rho_1(v)+\rho_1^h(v)}=\frac{1}{1+e^{\frac{K_1(v)}{T}}}.$$

Comparing this with the Fermi-Dirac distribution we observe that $K_1(v)$ plays the role of an excitation energy measured from the Fermi level. Correspondingly, ϵ has the same interpretation. We return to this point in the next section.

We see from eqs (2.22),(2.23) and (2.24) that the functions $K_{n\geq 2}(v)$ are always positive while $K_1(v)$ and $\epsilon(\gamma)$ may have positive and negative regions. Let us now consider the case $T = 0$ in order to find the ground state of the system, i.e. the eigenstate of the t-J hamiltonian at arbitrary filling with an external magnetic field corresponding to the lowest eigenvalue. Taking the limit $T \to 0$ in (2.22),(2.23) and (2.24) we obtain the following expressions

$$K_{n\geq 2} = H$$

$$K_1(v) = -\frac{4}{1+v^2} + H - \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{2}{4+(v-k)^2} K_1^-(k) + \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{1}{1+(v-k)^2}\epsilon^-(k) \tag{2.27}$$

$$\epsilon(\gamma) = -\frac{H}{2} + A + \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{1}{1+(\gamma-k)^2} K_1^-(k) \tag{2.28}$$

where

$$K_1^-(v) = \begin{cases} K_1(v), & \text{for } K_1(v) < 0 \\ 0, & \text{for } K_1(v) \geq 0 \end{cases}, \tag{2.29}$$

$$\epsilon^-(\gamma) = \begin{cases} \epsilon(\gamma), & \text{for } \epsilon(\gamma) < 0 \\ 0, & \text{for } \epsilon(\gamma) \geq 0 \end{cases}. \tag{2.30}$$

In addition from eqs.(2.25) and (2.26) we have that at $T = 0$ the densities of roots ρ and σ are different from zero only in the regions where K and ϵ are negative (below the Fermi energy). Since $K_{n\geq 2}$ is always positive we conclude that the ground state involves only real roots

$$\begin{aligned} &\rho_{n\geq 2}(v) = 0 \\ &\rho_1(v) = 0, \quad if \quad K_1(v) \geq 0, \\ &\sigma(\gamma) = 0, \quad if \quad \epsilon(\gamma) \geq 0 \end{aligned} \tag{2.31}$$

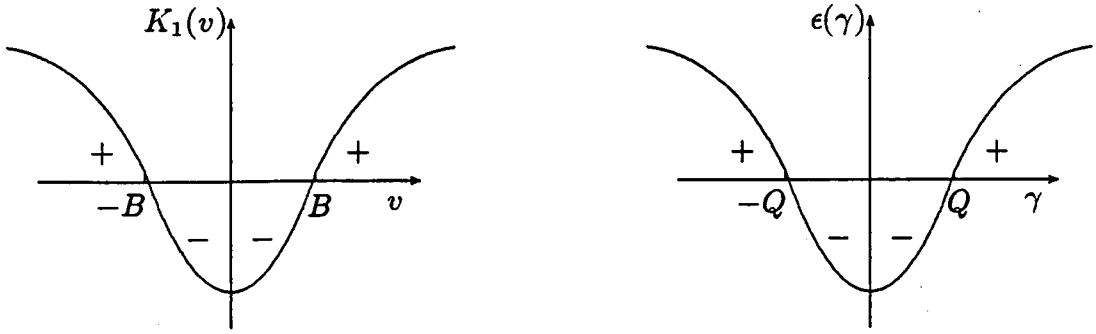
and states with $n \geq 2$ (complex strings) correspond to elementary excitations. We return to this point later.

Let us come back to the Yang's equations (2.27) and (2.28). We can verify that $K_1(v)$ and $\epsilon(\gamma)$ satisfy the following properties

$$K_1(-v) = K_1(v); \quad \epsilon(-\gamma) = \epsilon(\gamma) \tag{2.32}$$

$$\frac{dK_1(v)}{dv} > 0 \quad for \quad v > 0$$
$$\frac{d\epsilon(\gamma)}{d\gamma} > 0 \quad for \quad \gamma > 0$$

and change sign at some point, say $v = \pm B$ and $\gamma = \pm Q$. The figures below illustrate qualitatively the behavior of the functions $K_1(v)$ and $\epsilon(\gamma)$



The parameters B and Q correspond to "Fermi levels" , since in the ground state they delimit the region where the distributions of real roots $\rho_1(v)$ and $\sigma(\gamma)$ are non vanishing. They are determined by the requirement $K_1(B) = 0$ and $\epsilon(Q) = 0$ and depend on the magnetic field H and chemical potential A according to eqs.(2.27) and (2.28). In the next section we determine B and Q numerically as well as analytically for some limit cases. Now, we can rewrite the Yang's equations as

$$K_1(v) = -\frac{4}{1+v^2} + H - \frac{1}{\pi}\int_{-B}^{B} dk \frac{2}{4+(v-k)^2} K_1(k) + \frac{1}{\pi}\int_{-Q}^{Q} dk \frac{1}{1+(v-k)^2}\epsilon(k) \tag{2.33}$$

$$\epsilon(\gamma) = -\frac{H}{2} + A + \frac{1}{\pi}\int_{-B}^{B} dk \frac{1}{1+(\gamma-k)^2} K_1(k) \tag{2.34}$$

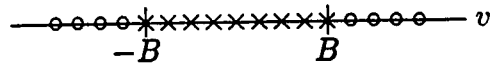
since $K_1^-(v) = K_1(v)$ for $|v| < B$ and $\epsilon^-(\gamma) = \epsilon(\gamma)$ for $|\gamma| < Q$. These equations play an essential role in the investigation of the elementary excitation spectrum of the supersysmmetric t-J model. They can be exactly solved for special limit values of B and Q using the Wiener-Hopf technique and small $B(Q)$ expansion.

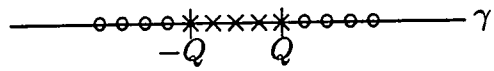
For later convenience we write down the relations for the magnetization (2.14) and filling (2.15) in terms of the quantities B and Q as

$$\frac{S_z}{L} = \frac{1}{2} - \int_{-B}^{B} dv \rho_1(v) + \frac{1}{2}\int_{-Q}^{Q} d\gamma \sigma(\gamma) \tag{2.35}$$

$$F = 1 - \int_{-Q}^{Q} d\gamma \sigma(\gamma) \tag{2.36}$$

At the end of this section we discuss qualitatively our results for the ground state of the system at arbitrary filling in the presence of an external magnetic field. The ground state is characterized by having only real roots in the v and γ axis, as illustrated in the figure below





At the half filling (one electron per site) there are no second level roots in the γ-axis ("Fermi level" Q is zero, according to eq.(2.36)). By removing electrons from the system the Fermi level Q increases continuously up to a situation where the filling is zero ($Q = \infty$). The Fermi level B associated with the first level roots depends basically on an applied magnetic field H similar to the case of the XXX Heisenberg model. A non zero value of H yields a finite value of B. The larger is the external magnetic field the smaller is the Fermi level B. In the next section we determine the Fermi levels B and Q as well as some physical quantities of interest such as magnetization or filling as a function of the chemical potential A and magnetic field H.

3. FERMI LEVELS, MAGNETIZATION AND FILLING

In this section we find the relation between the integration limits ("Fermi levels") B and Q and the magnetic field H and the chemical potential A for the cases *i)* system near the half filling in the absence of a magnetic field ; *ii)* system at the half filling in the presence of a magnetic field. The behavior of the filling as a function of the chemical potential as well as the relation between the magnetization and the magnetic field are also obtained for these cases.

The integration limits B and Q can be calculated through the equations

$$K_1(v) = -\frac{4}{1+v^2} + H - \frac{1}{\pi}\frac{2}{4+v^2} * K_1^-(v) + \frac{1}{\pi}\frac{1}{1+v^2} * \epsilon^-(v), \tag{3.1}$$

$$\epsilon(\gamma) = -\frac{H}{2} + A + \frac{1}{\pi}\frac{1}{1+\gamma^2} * K_1^-(\gamma), \tag{3.2}$$

introduced in the last section (see eqs.(2.27) and (2.28)) together with the notation

$$K_1 = K_1^+ + K_1^- = \begin{cases} K_1^-, & \text{for } K_1 < 0 \\ K_1^+, & \text{for } K_1 \geq 0 \end{cases}, \tag{3.3}$$

and

$$\epsilon = \epsilon^+ + \epsilon^- = \begin{cases} \epsilon^-, & \text{for } \epsilon < 0 \\ \epsilon^+, & \text{for } \epsilon \geq 0 \end{cases}. \tag{3.4}$$

We begin by Fourier transforming (3.1) and (3.2) (see Appendix B for further details)

$$\tilde{K}_1(x) = -4\pi e^{-|x|} + 2\pi\delta(x)H - e^{-2|x|}\tilde{K}_1^-(x) + e^{-|x|}\tilde{\epsilon}^-(x), \tag{3.5}$$

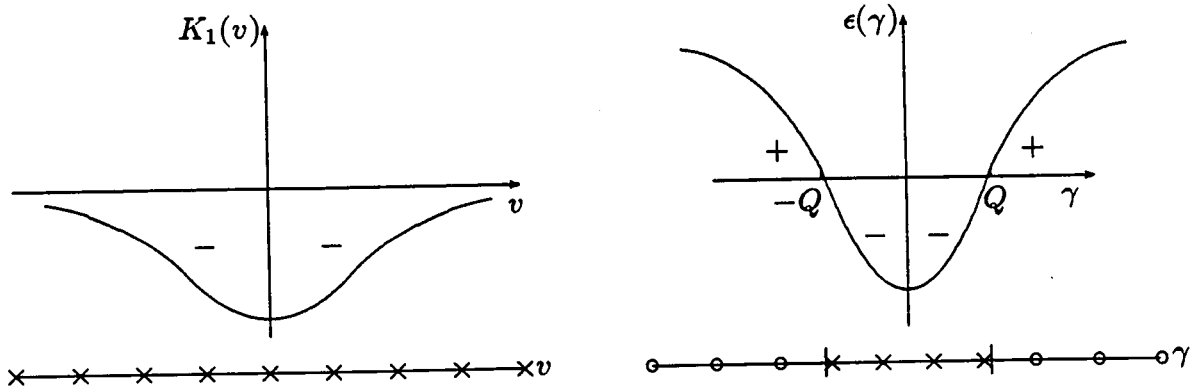
$$\tilde{\epsilon}(x) = -2\pi\delta(x)\frac{H}{2} + 2\pi\delta(x)A + e^{-|x|}\tilde{K}_1^-(x). \tag{3.6}$$

One may replace $\tilde{K}_1^-(\tilde{\epsilon}^-)$ by $\tilde{K}_1^- = \tilde{K}_1 - \tilde{K}_1^+$ ($\tilde{\epsilon}^- = \tilde{\epsilon} - \tilde{\epsilon}^+$). Depending on the physical situation we will write eqs.(3.5)-(3.6) in terms of $\tilde{K}_1^+$ ($\tilde{\epsilon}^+$) or $\tilde{K}_1^-$ ($\tilde{\epsilon}^-$).

3.1 THE FERMI LEVEL Q, FILLING AND CHEMICAL POTENTIAL

Consider the case where no external magnetic field is applied ($H = 0$) on the system. From eq. (3.1) we have that K_1 is always negative, which means $B = \infty$. Furthermore, from (2.36) we see that the larger is the values of Q the smaller is the filling. This situation is depicted below through the graphs for K_1, ϵ and ground state

configuration.



Our task is to find the relation between the "Fermi level" Q and the chemical potential A (or the filling F) for small and large values of Q. From eq.(3.5) we express $\tilde{K}_1^-$ as a function of $\tilde{K}_1^+$ and $\tilde{\epsilon}^-$ and insert it into eq.(3.6). Using the fact that $\tilde{K}_1^+ = 0$ we obtain

$$\tilde{\epsilon}(x) = 2\pi\delta(x)A - 4\pi\tilde{G}_1(x) + \tilde{G}_1(x)\tilde{\epsilon}^-, \tag{3.7}$$

with

$$\tilde{G}_l(x) = \frac{e^{-l|x|}}{2\cosh x}, \quad l = integer \tag{3.8}$$

For example, for $l = 0$ and $l = 1$ we have

$$G_0(v) = \frac{1}{4}\frac{1}{\cosh(\frac{v\pi}{2})},$$

$$G_1(v) = \frac{1}{4\pi}\left(\beta(1 - i\frac{v}{2}) + \beta(1 + i\frac{v}{2})\right)$$

where $\beta(x)$ is the beta function [59]. We solve eq.(3.7) separately for small and large values of Q.

a) SYSTEM NEAR HALF FILLING (SMALL Q)

Transforming back eq. (3.7) and using (3.4) we find

$$\epsilon(\gamma) = A - 4\pi G_1(\gamma) + \int_{-Q}^{Q} dk G_1(\gamma - k)\epsilon(k). \tag{3.9}$$

For very small values of Q we can approximate the integral above by its integrand at the mean value of the integration limits ($k = 0$) multiplied by the interval of integration.

$$\epsilon(\gamma) \simeq A - 4\pi G_1(\gamma) + 2QG_1(\gamma)\epsilon(0). \tag{3.10}$$

The value of $\epsilon(0)$ is obtained directly from the equation above by setting $\gamma = 0$. Then, after some rearrangements we find

$$\epsilon(\gamma) \simeq \frac{A - 4\pi G_1(\gamma) + 2QA\Big(G_1(\gamma) - G_1(0)\Big)}{1 - 2QG_1(0)}. \tag{3.11}$$

The condition $\epsilon(Q) = 0$ determines Q. Expanding $G_1(\gamma)$ for small Q values we have

$$G_1(Q) \simeq G_1(0) - \frac{3}{32\pi}\zeta(3)Q^2; \quad G_1(0) = \frac{\ln 2}{2\pi} \tag{3.12}$$

and using (3.11) we obtain the relation between the chemical potential A and the "Fermi level" Q

$$Q = \sqrt{\frac{8}{3\zeta(3)}(2\ln 2 - A)}, \tag{3.13}$$

for small Q. Here $\zeta(z)$ is the zeta Riemann function [59]. From the foregoing expression we see that there is a critical chemical potential $A_c = 2\ln 2 \geq A$ such that the equality characterizes a system at the half filling ($F = 1$). We also derive the relation between the filling F and Q by using a similar procedure involving eqs.(2.11), (2.12), (2.31) and (2.36)

$$F \simeq 1 - \frac{\ln 2}{\pi}Q. \tag{3.14}$$

Then, combining (3.13) with (3.14) we get the relation between the filling F and the chemical potential A for the case that the system is near the half filling.

$$F \simeq 1 - \frac{\ln 2}{\pi}\sqrt{\frac{8}{3\zeta(3)}(2\ln 2 - A)}. \tag{3.15}$$

Next we analyze the limit for very large Q values.

b) SYSTEM NEAR ZERO FILLING (LARGE Q)

In this case is more convenient to work with $\tilde{\epsilon}^+$ instead of $\tilde{\epsilon}^-$ in eq. (3.7). Then using eq.(3.4) and transforming the result we get

$$\epsilon(\gamma) = -\frac{4}{\pi}\frac{2}{4+\gamma^2} - 2A - \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{2}{4+(\gamma-k)^2}\epsilon^+(k). \tag{3.16}$$

This equation is linear in ϵ. It is convenient to separate $\epsilon = \epsilon_a + \epsilon_b$ with

$$\epsilon_a(\gamma) = -\frac{4}{\pi}\frac{2}{4+\gamma^2} - \frac{1}{\pi}\int_{-\infty}^{-Q} dk \frac{2}{4+(\gamma-k)^2}\epsilon_a(k) - \frac{1}{\pi}\int_{Q}^{\infty} dk \frac{2}{4+(\gamma-k)^2}\epsilon_a(k) \tag{3.17}$$

$$\epsilon_b(\gamma) = -2A - \frac{1}{\pi}\int_{-\infty}^{-Q} dk \frac{2}{4+(\gamma-k)^2}\epsilon_b(k) - \frac{1}{\pi}\int_{Q}^{\infty} dk \frac{2}{4+(\gamma-k)^2}\epsilon_b(k). \quad (3.18)$$

The procedure to obtain ϵ_a and ϵ_b is to rewrite these equations in the Wiener-Hopf form. This can be achieved by changing variables $\gamma \to \gamma + Q$ and defining $y(\gamma) = \epsilon(\gamma + Q)$. After some manipulations we find

$$y_a(\gamma) = -\frac{4}{\pi}\frac{2}{4+(\gamma+Q)^2} - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma-k)^2} y_a(k) - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma+k+2Q)^2} y_a(k), \quad (3.19)$$

$$y_b(\gamma) = -2A - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma-k)^2} y_b(k) - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma+k+2Q)^2} y_b(k) \quad (3.20)$$

Assuming large Q (very small chemical potential A or filling F) we can verify that the last term in eqs.(3.19) and (3.20) is of the order of $1/Q^2$. Thus, in order to obtain the leading term for Q tending to infinity we may neglect these expressions. Of course, if we are interested in corrections to the leading contribution we may take these terms into account and solve the equations (3.19) and (3.20) by iteration [57]. For our purpose it is sufficient to consider

$$y_a^{(1)}(\gamma) = -\frac{4}{\pi}\frac{2}{4+(\gamma+Q)^2} - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma-k)^2} y_a^{(1)}(k), \quad (3.21)$$

$$y_b^{(1)}(\gamma) = -2A - \frac{1}{\pi}\int_0^{\infty} dk \frac{2}{4+(\gamma-k)^2} y_b^{(1)}(k), \quad (3.22)$$

where the superscript (1) means the leading contribution to y. These equations are of the Wiener-Hopf kind and can be solved analytically. In fact, this method allows for a solution of integral equations in a semi-infinity domain through the use of Fourier transform (see Appendix B). We begin with eq. (3.21), which can be expressed in the form

$$y_a^{(1)}(\gamma) = -\frac{4}{\pi}\frac{2}{4+(\gamma+Q)^2} + \frac{1}{\pi}\int_{-\infty}^{\infty} dk \frac{2}{4+(\gamma-k)^2} y_{a+}^{(1)}(k), \quad (3.23)$$

where $y_{a+}^{(1)}(k) = y_a^{(1)}(k)$ for $k > 0$ and zero otherwise. Its Fourier transform is given by (see Appendix B)

$$\tilde{y}_{a+}^{(1)}(x) = \frac{1}{2\pi i} g_+(x) \int_{-\infty}^{\infty} dz \frac{4e^{-izQ}e^{-2|z|}}{z-x-i\eta} g_-(z), \quad (3.24)$$

where η is a positive infinitesimal and g_+ and g_- are analytic functions in the upper and lower half complex plane, respectively

$$g_+(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{\eta - ix}{\pi e} \right)^{\frac{ix}{\pi}} \Gamma\left(\frac{1}{2} - \frac{ix}{\pi} \right), \tag{3.25}$$

$$g_-(x) = \frac{1}{\sqrt{2\pi}} \left(\frac{\eta + ix}{\pi e} \right)^{-\frac{ix}{\pi}} \Gamma\left(\frac{1}{2} + \frac{ix}{\pi} \right), \tag{3.26}$$

and they factorize the function

$$1 - \mathcal{F}\left(-\frac{1}{\pi} \frac{2}{4+\gamma^2} \right) = 1 + e^{-2|x|} = \frac{1}{g_+(x) g_-(x)} \tag{3.27}$$

Here Γ is the gamma function [59] and $\mathcal{F}$ means Fourier transformation. Correspondingly, from eq.(3.22) we can also obtain the Fourier transform of $y^{(1)}_{b+}(\gamma)$

$$\tilde{y}^{(1)}_{b+}(x) = -\frac{1}{2\pi i} g_+(x) \int_{-\infty}^{\infty} dz \frac{4\pi A \delta(z)}{z - x - i\eta} g_-(z). \tag{3.28}$$

The condition $\epsilon(Q) = 0$ determines Q. Using that

$$\epsilon^+(Q) = \lim_{x \to \infty} -ix\tilde{y}_+(x) \tag{3.29}$$

together with $\tilde{y}^{(1)}_+(x) = \tilde{y}^{(1)}_{a+}(x) + \tilde{y}^{(1)}_{b+}(x)$, where $\tilde{y}^{(1)}_a(x)$ and $\tilde{y}^{(1)}_b(x)$ are given by (3.24) and (3.28) we obtain the relation between Q and the chemical potential A (for small Q)

$$Q \simeq \frac{2}{\sqrt{A}} \tag{3.30}$$

We can also derive the relation between the filling F and the Fermi level Q through eqs.(2.11), (2.12) ,(2.31) and (2.36) by a similar procedure,

$$F \simeq \frac{4}{\pi Q}. \tag{3.31}$$

Therefore, for a system near zero filling, the relation between F and A is given by

$$F \simeq \frac{2\sqrt{A}}{\pi}. \tag{3.32}$$

Hence, the value $A = 0$ corresponds to a situation where all the sites of the chain are not occupied (zero filling).

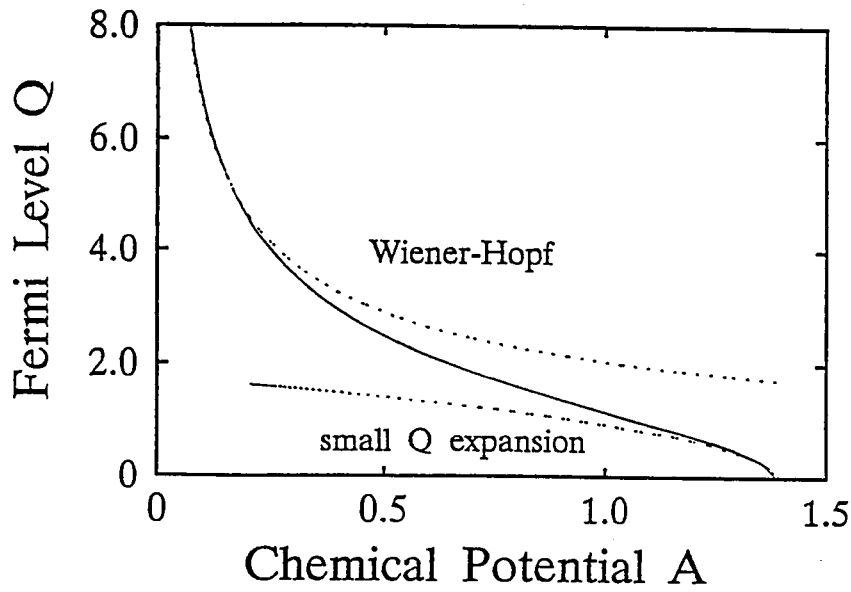


Fig.1: *Numerical and analytical curves for the Fermi level Q as a function of the chemical potential A*

Fig. 1 shows the "Fermi level" Q as a function of the chemical potential A calculated from the approximated formulas (3.13) and (3.30). The exact solution obtained from a numerical integration of eqs.(3.1) and (3.2) is also displayed.

We can see that the results derived from the Wiener-Hopf method and from the mean value theorem agree quite well with the numerical calculation for large and small values of Q, respectively. Although only the leading term for large values of Q was considered so far we see that the Wiener-Hopf method yields reasonable results even for intermediate values of A. The relation between the filling F and the chemical potential A is illustrated in the fig. 2.

The numerical curve is obtained from eqs.(2.11), (2.12) and (2.36) and the analytical ones from eqs.(3.15) and (3.32). In this case we also observe a good agreement between the analytical approximations and the exact result. The zero filling corresponds to $A = 0$ while the half-filling is characterized by $A = 2\ln 2$.

3.2 FERMI LEVEL B, MAGNETIZATION AND MAGNETIC FIELD

We consider the case that the system is at half filling, i.e, $Q = 0$, as can be seen from eq.(2.36). In addition, from (2.33) and (2.35) we have that the smaller is the magnetic field the larger is the value of the integration limit B. This situation is exemplified

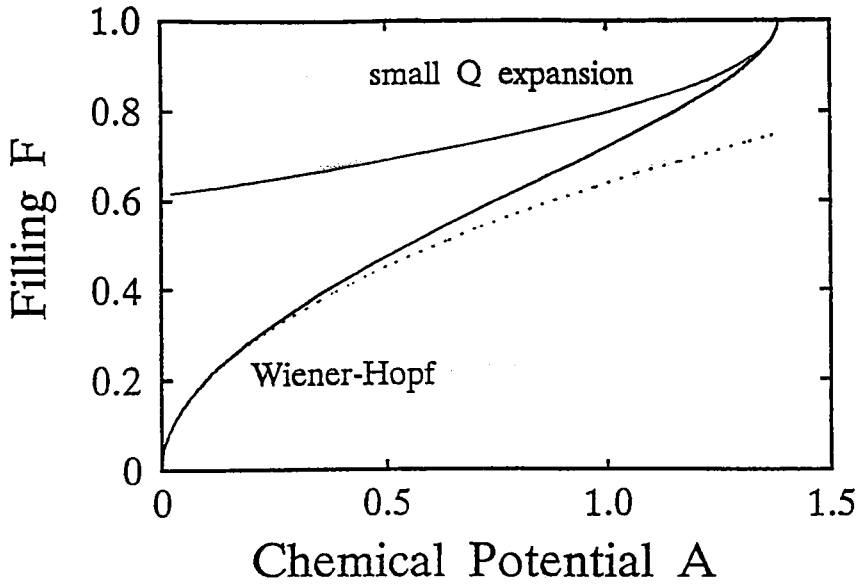
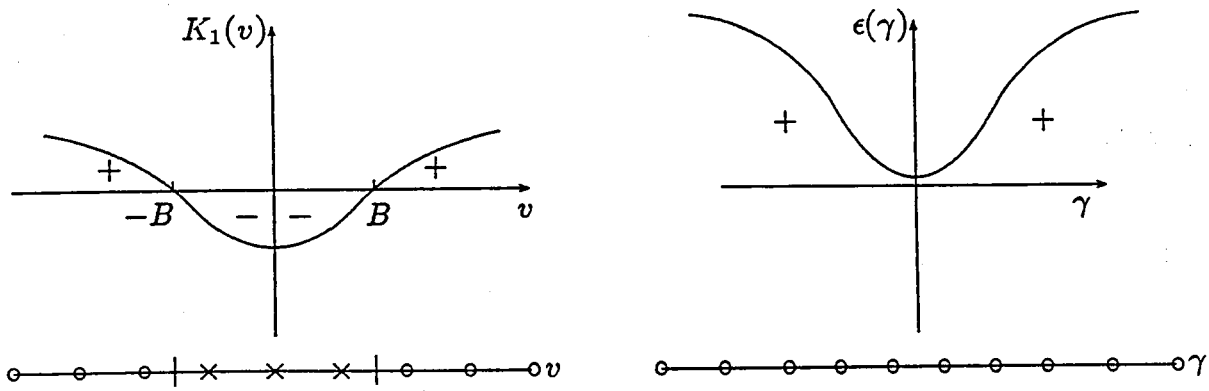


Fig.2: *Numerical and analytical results for the Filling F versus the chemical potential A*

below in the graphics of $K_1(v)$ and $\epsilon(\gamma)$ and the ground state configuration



Our purpose is to obtain the relation between the Fermi level B and the magnetic field H for small and large values of B.

a) SMALL MAGNETIC FIELD (LARGE B)

We start rewriting the eq.(3.5) in terms of $\tilde{K}^+$ instead of using $\tilde{K}^-$. Then, anti Fourier transforming the result we have (with G_0 and G_1 given by eq.(3.8))

$$K_1(v) = -4\pi G_0(v) + \frac{H}{2} + \int_{-\infty}^{-B} dk G_1(v-k)K_1(k) + \int_{B}^{\infty} dk G_1(v-k)K_1(k). \quad (3.33)$$

As in the previous case, invoking linearity of the equation above in ϵ, we may separate it using $K_1 = K_{1a} + K_{1b}$, where $K_{1a}(K_{1b})$ is the solution associated with the

inhomogeneous term $\frac{H}{2}\left(-4\pi G_0(v)\right)$. Changing the variable $v \to v + B$ and defining $y(v) \equiv K_1(v + B)$ we obtain, after some rearrangements

$$\begin{aligned} y_a(v) &= \frac{H}{2} + \int_0^\infty dk G_1(v-k) y_a(k) + \int_0^\infty dk G_1(v+2B+k) y_a(k) \\ y_b(v) &= -4\pi G_0(v+B) + \int_0^\infty dk G_1(v-k) y_b(k) + \int_0^\infty dk G_1(v+2B+k) y_b(k) \end{aligned} , \tag{3.34}$$

For large B we can neglect the last term in the expressions above and apply the Wiener-Hopf method to solve the integral equations. The procedure is essentially the same as in the previous case for large values of Q. Thus, we find

$$\begin{aligned} \tilde{y}_+^{(1)}(x) &= \tilde{y}_{a+}^{(1)}(x) + \tilde{y}_{b+}^{(1)}(x) \\ &= i\frac{H}{2}\frac{1}{g_-(0)}\frac{1}{g_+(x)}\frac{1}{x+i\eta} - 2\pi i e^{-\frac{\pi B}{2}}\frac{1}{g_-(-\frac{i\pi}{2})}\frac{1}{g_+(x)}\frac{1}{x+i\frac{\pi}{2}} . \end{aligned} \tag{3.35}$$

Here η is a positive infinitesimal and g_+ and g_- are given by eqs.(3.25) and (3.26). The condition $K_1(B) = 0$, which is equivalent to $\lim_{x\to\infty} -ix\tilde{y}_+(x)$, establishes the relation between B and H

$$B \simeq -\frac{2}{\pi}\ln\left(\frac{H}{2\pi}\sqrt{\frac{e}{2\pi}}\right), \tag{3.36}$$

for large B. This is the same relation obtained for the Heisenberg model H_{XXX} (see ref.[57]), since at the half filling both Yang equations are similar. Using the same method we derive the relation between the magnetization S_z and the Fermi level B. Consequently, for small magnetic field H, the relation between S_z and H is

$$S_z = \frac{H}{\pi e}. \tag{3.37}$$

b) LARGE MAGNETIC FIELD (SMALL B)

Consider the case where B is small. Using that $\epsilon^- = 0$ and $K_1^-(v) = K_1(v)$ for $|v| < B$ in eq. (3.1) we find

$$K_1(v) = -\frac{4}{1+v^2} + H - \frac{1}{\pi}\int_{-B}^{B} dk \frac{2}{4+(v-k)^2} K_1(k). \tag{3.38}$$

Then proceeding along the same lines as in the calculation of $Q \times A$ for small Q values and using the condition $K^-(B) = 0$ we obtain the relation between the magnetic field H and the Fermi level B

$$B \simeq \sqrt{\frac{4-H}{4}} \tag{3.39}$$

and also the relation between H and the magnetization S_z

$$\frac{S_z}{L} \simeq \frac{1}{2} - \frac{2}{\pi}\sqrt{\frac{4-H}{H}}, \tag{3.40}$$

for large H values. We see in these expressions that there is a finite critical magnetic field ($H_c = 4$) for which $B = 0$ and $\frac{S_z}{L} = \frac{1}{2}$. This means a ferromagnetic transition (the number (N) of down spins plus holes vanishes).

In the figs. 3 and 4 we illustrate the behavior of the Fermi level B and the magnetization S_z versus the magnetic field H.

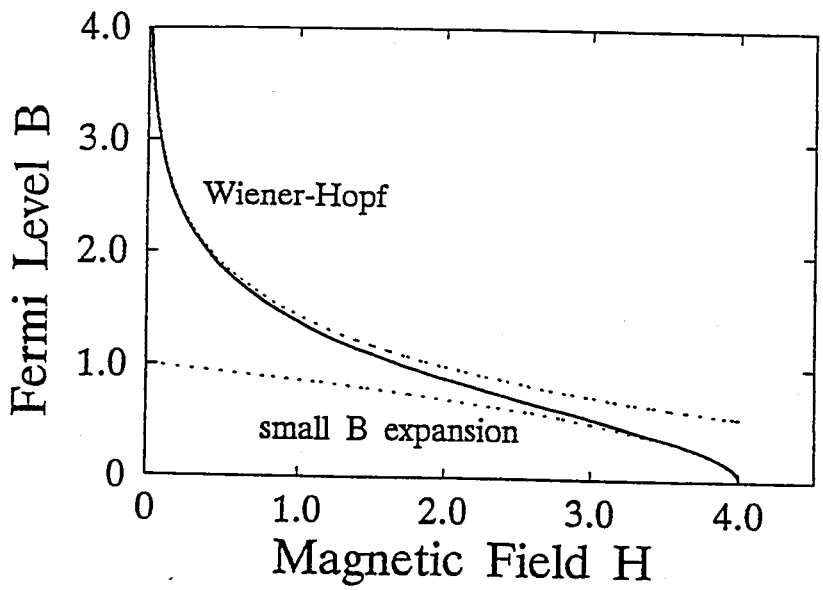


Fig.3: *Numerical and analytical results for the Fermi level B as a function of the magnetic field H*

The numerical curves are obtained from eqs.(3.1) , (3.2), (2.11), (2.12) and (2.35). We see a very good agreement between them and our analytical results. For low magnetic field the magnetization vanishes linearly, while for $H = H_c = 4$ the system becomes ferromagnetic.

4. GROUND STATE AND ELEMENTARY EXCITATIONS

In this section we derive the energy and momentum for the ground state of the system at half filling. This is the simplest case and can be solved just using Fourier

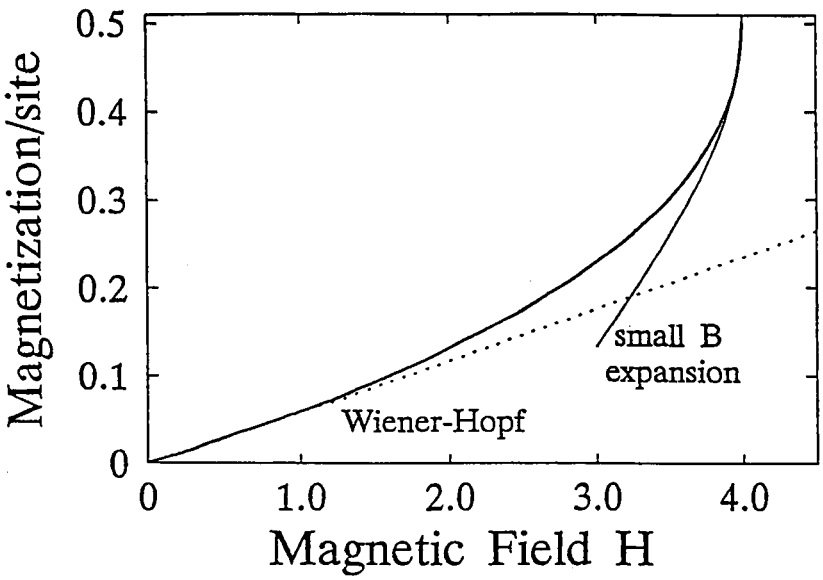


Fig.4: *Numerical and analytical results for* $S_z \times H$

transformations. Furthermore, we compute these physical quantities at half-filling for some elementary excitations as spinons, holons, .. and give the corresponding dispersion relations. We follow the basic ideas of Faddeev and Takhtajan [23,50] who studied the ground state and excitations for the isotropic Heisenberg model in the antiferromagnetic regime. Our case is more complex due to the nested Bethe ansatz with two levels, which originates new type of excitations. This subject was discussed qualitatively in section 5 of chapter 2. The case of the system at arbitrary filling is also discussed. By performing numerical calculations we present the dispersion relations of the holon-antiholon excitation.

The energy and momentum per lattice site can be calculated through the equations

$$\frac{E}{L} = 1 - \sum_n \int_{-\infty}^{\infty} dv \frac{4n}{n^2+v^2} \rho_n(v), \tag{4.1}$$

$$\frac{P}{L} = \sum_n \int_{-\infty}^{\infty} dv \left(-2 \arctan v + \pi\right) \rho_n(v), \tag{4.2}$$

introduced in section 3 of the previous chapter together with the equation for the density of roots ρ_1 (eq.(2.11))

$$\rho_1(v) + \rho_1^h(v) = \frac{1}{\pi}\frac{1}{1+v^2} - \frac{1}{\pi}\frac{2}{4+v^2} * \rho_1(v) + \frac{1}{\pi}\frac{1}{1+v^2} * \sigma(v). \tag{4.3}$$

By Fourier transforming this equation we get

$$\tilde{\rho}_1(x)\left(1 + e^{-2|x|}\right) = -\tilde{\rho}_1^h(x) + e^{-|x|} + e^{-|x|}\tilde{\sigma}(x). \tag{4.4}$$

We can solve this equation according to the physical situation under consideration. We recall that all cases that will be treated here were already discussed qualitatively in section 5 of chapter 2. We begin with the case of the ground state.

4.1) GROUND STATE

By minimizing the free energy we found that the ground state for half-filling $F = 1$ involves only first level real roots. This situation corresponds to the configuration

$$\rho_{n\geq 2} = 0, \quad \sigma = 0, \quad \rho_n^h = 0. \tag{4.5}$$

Therefore, from eq.(4.4) we find that the Fourier transform of the density of first level real roots is

$$\tilde{\rho}_1^{GS} = \frac{1}{2\cosh x}. \tag{4.6}$$

Then, the corresponding energy (4.1) and momentum (4.2) have the form

$$\frac{E^{GS}}{L} = 1 - 2\ln 2, \tag{4.7}$$

$$\frac{P^{GS}}{L} = \frac{\pi}{2}. \tag{4.8}$$

As shown in the last chapter, the magnetization of the ground state is $S_z = 0$. This result can also be obtained using eqs.(4.6) and (2.14).

Next we consider some elementary excitations at half-filling.

4.2) "SPINONS"

This excitation is obtained by removing a real root from the I^1-axis or introducing a first level BA-hole, which corresponds to a spin flip. In this case we have

$$\rho_{n\geq 2} = 0, \quad \sigma = 0, \quad \rho_1^h = \frac{1}{L}\delta(v - v_h), \tag{4.9}$$

where v_h is the position of a first level BA-hole. We recall that a one-spinon state exists on lattices with an odd number of lattice sites, otherwise spinons appear pairwise. From eq.(4.4) we find the form of the Fourier transform of the density of first level roots

$$\tilde{\rho}_1(x) = \tilde{\rho}_1^{GS}(x) - \frac{\tilde{\rho}_1^h(x)}{1 + e^{-2|x|}}. \tag{4.10}$$

The energy and momentum of this state measured from the ground state have the form

$$\Delta E(v_h) = E(v_h) - E^{GS} = \frac{\pi}{\cosh\left(\frac{\pi v_h}{2}\right)}, \tag{4.11}$$

$$\Delta P(v_h) = P(v_h) - P^{GS} = \arctan\left(\sinh\left(\frac{\pi v_h}{2}\right)\right) - \frac{\pi}{2}, \tag{4.12}$$

where $E(v_h)$, $P(v_h)$ are calculated from eqs.(4.1),(4.2) and (4.10) and E^{GS},P^{GS} are given by (4.7) and (4.8). For vanishing magnetic field we find from eq.(3.1) or (3.33) $K_1(v_h) = -4\pi G_0(v_h) = -\Delta E(v_h)$ (see eq.(3.8)) which confirms the fact that the Yang function K_1 measures the energy excitation. By introducing more than one first-level BA-hole we can verify the additivity of the contribution of them in the energy and momentum equations. From eqs.(4.11) and (4.12) we have the following dispersion relation

$$\Delta E = \pi \sin(\Delta P), \quad 0 \leq \Delta P \leq \pi. \tag{4.13}$$

This is a gapless excitation and for small momenta, the relation between energy and momentum turns out to be linear. The magnetization of this state is $\frac{S_z}{L} = \frac{1}{2}$ (see eqs.(2.14) and (4.10)).

Another elementary excitation is a string in the v axis.

4.3) STRING EXCITATION

The presence of a 2-string in the v-axis characterizes another elementary excitation, where

$$\rho_{n\geq 3} = 0, \quad \sigma = 0, \quad \rho_1^h = \frac{1}{L}\sum_{h=1}^{2}\delta(v - v_h), \quad \rho_2(v) = \frac{1}{L}\delta(v - v_s). \tag{4.14}$$

Here, v_h is a position of the first-level BA-hole and v_s is the position of the 2-string. The magnetization of this excitation is $S_z = 0$ (see section 5 of chapter 2). Then, from eq.(4.4) we get the Fourier transform of the density of first-level real roots,

$$\tilde{\rho}_1(x) = \tilde{\rho}_1^{GS}(x) - \frac{\tilde{\rho}_1^h(x)}{1 + e^{-2|x|}} - \tilde{\rho}_2(x)e^{-|x|} \tag{4.15}$$

From eqs.(4.1) and (4.2) we obtain the expressions for the energy and momentum of this excitation, which have the same form as those calculated in the previous example. This can be easily seen by the fact that the contribution of the last term in eq.(4.15)

for the energy and momentum cancels the contribution of $n = 2$ in eqs.(4.1) and (4.2). The only difference between theses states and the previous ones is the value of the magnetization. This result can also be proven by similar arguments for higher-strings. Another excitation of interest are the "holons".

4.4) "HOLONS"

This excitation is obtained by filling a vacancy in the γ-axis, which means physically, removing one electron of the system. In this situation we have

$$\rho_{n\geq 2} = 0, \quad \sigma = \frac{1}{L}\delta(\gamma - \gamma_r), \tag{4.16}$$

where γ_r is the position of a root in the γ-axis. It is pointed out that this configuration exists for a lattice with an odd number of sites, otherwise there would appear BA-holes in the I^1-axis. From eq.(4.4) we obtain

$$\tilde{\rho}_1(x) = \tilde{\rho}_1^{GS}(x) - \frac{e^{-|x|}\tilde{\sigma}(x)}{1 + e^{-2|x|}}. \tag{4.17}$$

Then, we have the following expressions for the energy and momentum measured from the ground state

$$\begin{aligned}\Delta E(\gamma_r) = E(\gamma_r) - E^{GS} =& 4\int_0^\infty dx \frac{\cos(\gamma_r x)}{e^{2|x|}+1} \\ =& \beta\left(1 - i\frac{\gamma_r}{2}\right) + \beta\left(1 + i\frac{\gamma_r}{2}\right)\end{aligned} \tag{4.18}$$

$$\begin{aligned}\Delta P(\gamma_r) = P(\gamma_r) - P^{GS} =& 2\int_0^\infty \frac{dx}{x}\frac{\sin(\gamma_r x)}{e^{2|x|}+1} \\ =& \frac{1}{2}\int_0^{\gamma_r} d\gamma \left(\beta\left(1 - i\frac{\gamma}{2}\right) + \beta\left(1 + i\frac{\gamma}{2}\right)\right)\end{aligned} \tag{4.19}$$

where $\beta(x)$ is the beta function [59]. $E(\gamma_r)$, $P(\gamma_r)$ are calculated from eqs.(4.1),(4.2) and (4.17) and E^{GS}, P^{GS} are given by (4.7) and (4.8). By means of eqs (3.9) and (3.8) we can show that $\Delta E(\gamma_r) - A = -\epsilon(\gamma_r)$, which means that the Yang function ϵ can be interpreted as the holon excitation energy measured from the Fermi level. For low excitation energies γ_r is close to zero. In this case, we can expand the expressions above, resulting in

$$\Delta E = \sum_{n=0} a_n \gamma_r^{2n}, \tag{4.20}$$

$$\Delta P = \sum_{n=0} \frac{a_n}{2(2n+1)} \gamma_r^{2n+1}, \tag{4.21}$$

where the coefficient a_n is given by

$$a_n = \begin{cases} 2\ln 2, & \text{for } n = 0 \\ \frac{(-1)^n(1-2^{-2n})\zeta(2n+1)}{2^{2n-1}}, & \text{otherwise}, \end{cases} \tag{4.22}$$

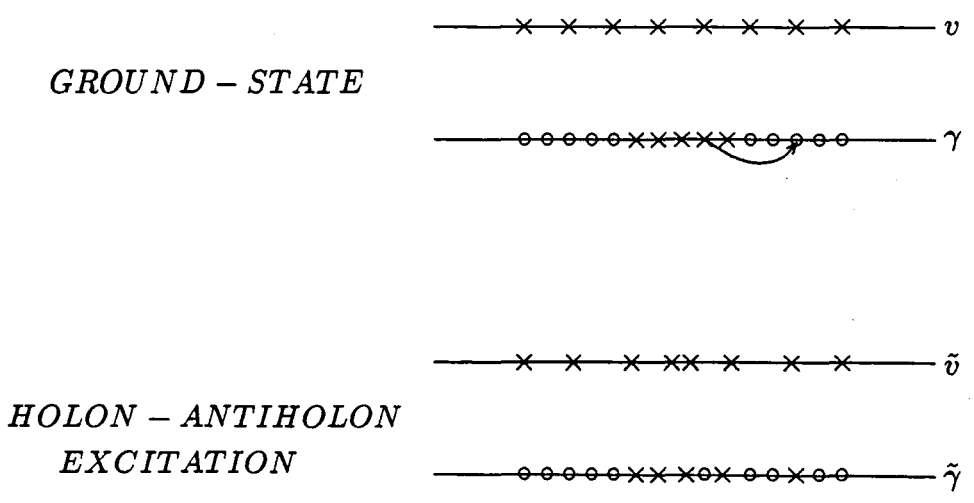
and ζ is the Riemann's zeta function [59]. Taking just the first terms in (4.20) and (4.21) we see that the relation between the energy and momentum is quadratic for small values of ΔP, i.e.,

$$\Delta E = \simeq a_0 + \frac{4a_1}{a_0^2}(\Delta P)^2 \tag{4.23}$$

This behavior can also be found by numerical calculations. The result above means that for half-filling holon excitations show an energy gap a_0 and an "effective mass" $\frac{a_0^2}{8a_1}$ contrasting the case of non half-filling discussed now.

HOLON-ANTIHOLON EXCITATION

In the case that the system is at arbitrary filling a non-trivial Fermi level Q appears (see section 3). Hence, there exists an excitation that does not change the number of particles. This excitation, the so called holon-antiholon excitation, involves the transfer of a second-level real root to a previously unoccupied state above the Fermi level Q. This means that the excitation is characterized by a second-level BA-hole (antiholon) γ_h with $|\gamma_h| < Q$ and a second-level real root (holon) γ_p with $|\gamma_p| > Q$. The situation is illustrated in the figure below



Notice that due to the presence of γ_p and γ_h the value of the other γ and v are changed (polarization effect) : $\gamma \to \tilde{\gamma}$, $v \to \tilde{v}$ so that the BAE eqs. ((3.44) and (3.45) of chapter 2) can be rewritten as

$$L\theta(\tilde{v}_\alpha) - \sum_{\beta=1}^{N} \theta_{11}(\tilde{v}_\alpha - \tilde{v}_\beta) + \sum_{\beta=1}^{M} \theta(\tilde{v}_\alpha - \tilde{\gamma}_\beta) - \theta(\tilde{v}_\alpha - \gamma_h) + \theta(\tilde{v}_\alpha - \gamma_p) = 2\pi \tilde{I}_\alpha \quad (5.1)$$

$$\sum_{\alpha=1}^{N} \theta(\tilde{v}_\alpha - \tilde{\gamma}_\beta) = 2\pi \tilde{J}_\beta \quad (5.2)$$

If we subtract these equations from eqs.(3.44),(3.45) and consider the variation $\tilde{v}_\alpha = v_\alpha - \Delta\tilde{v}_\alpha$, $\tilde{\gamma}_\beta = \gamma_\beta - \Delta\tilde{\gamma}_\beta$, which implies in first order that $\theta(\tilde{v}_\alpha) - \theta(v_\alpha) = \theta'(\tilde{v}_\alpha)\Delta\tilde{v}_\alpha$ we arrive at

$$\begin{aligned} L\theta'(\tilde{v}_\alpha) - \sum_{\beta=1}^{N} \theta'_{11}(\tilde{v}_\alpha - \tilde{v}_\beta)(\Delta\tilde{v}_\alpha) - \Delta\tilde{v}_\beta) + \sum_{\beta=1}^{M} \theta'(\tilde{v}_\alpha - \tilde{\gamma}_\beta)(\Delta\tilde{v}_\alpha - \Delta\tilde{\gamma}_\beta) \\ + \theta(\tilde{v}_\alpha - \gamma_h) - \theta(\tilde{v}_\alpha - \gamma_p) = 0 \end{aligned}, \quad (5.3)$$

$$\sum_{\alpha=1}^{N} \theta'(\tilde{v}_\alpha - \tilde{\gamma}_\beta)(\Delta\tilde{v}_\alpha - \Delta\tilde{\gamma}_\beta) = 0. \quad (5.4)$$

In the thermodynamic limit ($L \to \infty$ such that $\frac{N}{L} = const.$ and $\frac{M}{L} = const.$) we have

$$\begin{aligned} &\frac{\tilde{I}_\alpha}{L} \to x, \quad \tilde{v}_\alpha \to v(x), \quad \rho(v) = \frac{1}{L}\frac{dx(v)}{dv}, \\ &\frac{\tilde{J}_\beta}{L} \to y, \quad \tilde{\gamma}_\beta \to \gamma(x), \quad \sigma(\gamma) = \frac{1}{L}\frac{dy(\gamma)}{d\gamma}. \end{aligned}$$

In addition, the sums in eqs.(5.3) and (5.4) can be replaced by integrals. A rigorous proof may be found in Yang-Yang [60] Then, we have

$$\begin{aligned} &L\Delta v \underbrace{\left(\theta'(v) - \int_{-B}^{B} dv' \theta'_{11}(v - v')\rho^{GS}(v') + \int_{-Q}^{Q} d\gamma \theta'(v - \gamma)\sigma^{GS}(\gamma)\right)} \\ &+ \int_{-B}^{B} dv' \theta'_{11}(v - v') L\Delta v \rho^{GS}(v') - \int_{-Q}^{Q} d\gamma \theta'(v - \gamma) L\Delta\gamma \sigma^{GS}(\gamma) \\ &+ \theta(v - \gamma_h) - \theta(v - \gamma_p) = 0 \end{aligned} \quad (5.5)$$

$$L\Delta\gamma \underbrace{\int_{-B}^{B} dv \theta'(v - \gamma)\rho^{GS}(v)} - \int_{-B}^{B} dv \theta'(v - \gamma) L\Delta v \rho^{GS}(v) = 0 \quad (5.6)$$

The expressions under brackets can be recognized as the densities $2\pi\rho^{GS}(v)$ and $2\pi\sigma^{GS}(\gamma)$, respectively (see eqs.(2.11), (2.12) of this chapter and (3.46) of chapter 2). Now, let us introduce the shift functions

$$F_\rho(v|\gamma_p,\gamma_h) \equiv L\Delta v\rho^{GS}(v), \tag{5.7}$$

$$F_\sigma(\gamma|\gamma_p,\gamma_h) \equiv L\Delta\gamma\sigma^{GS}(\gamma),$$

which measure the deviation of the distribution of real roots ρ and σ due to the presence of γ_p and γ_h. Then, eqs.(5.5) and (5.6) can be written as

$$\begin{aligned} &F_\rho(v|\gamma_p,\gamma_h)+\frac{1}{\pi}\int_{-B}^{B}dv'\frac{2}{4+(v-v')^2}F_\rho(v'|\gamma_p,\gamma_h) \\ &\quad-\frac{1}{\pi}\int_{-Q}^{Q}d\gamma\frac{1}{1+(v-\gamma)^2}F_\sigma(\gamma|\gamma_p,\gamma_h)=\frac{1}{\pi}\int_{\gamma_p}^{\gamma_h}dv'\frac{1}{1+(v-v')^2} \end{aligned} \tag{5.8}$$

$$F_\sigma(\gamma|\gamma_p,\gamma_h)-\frac{1}{\pi}\int_{-B}^{B}dv\frac{1}{1+(v-\gamma)^2}F_\rho(v|\gamma_p,\gamma_h)=0 \tag{5.9}$$

Now it is possible to calculate the energy and momentum for this excitation over the ground state (or vacuum). These quantities are obtained by adding the contributions due to "vacuum polarization" to the corresponding "bare quantities". We begin by computing the energy $\Delta E(\gamma_p,\gamma_h)$ which is given by subtracting the energy of the excited state from the ground-state energy (see eq.(3.50) of chapter 2)

$$\Delta E(\gamma_p,\gamma_h)=-\sum_{\alpha=1}^{N}\frac{4}{1+\tilde{v}_\alpha^2}+\sum_{\alpha=1}^{N}\frac{4}{1+v_\alpha^2}. \tag{5.10}$$

Proceeding along the same line as in the calculation of eq.(5.5), i.e., taking into account $\tilde{v}_\alpha = v_\alpha - \Delta\tilde{v}_\alpha$ and considering the thermodynamic limit we find

$$\Delta E(\gamma_p,\gamma_h)=-\int_{-B}^{B}dv\frac{8v}{(1+v^2)^2}F_\rho(v|\gamma_p,\gamma_h). \tag{5.11}$$

Analogously, the momentum $\Delta P(\gamma_p,\gamma_h)$ is equal to the momentum of the excited state minus the momentum of the ground state

$$\begin{aligned} \Delta P(\gamma_p,\gamma_h) &= -\sum_{\alpha=1}^{N}2\arctan\tilde{v}_\alpha+\sum_{\alpha=1}^{N}2\arctan v_\alpha \\ &= \int_{-B}^{B}dv\frac{2}{1+v^2}F_\rho(v|\gamma_p,\gamma_h) \end{aligned} \tag{5.12}$$

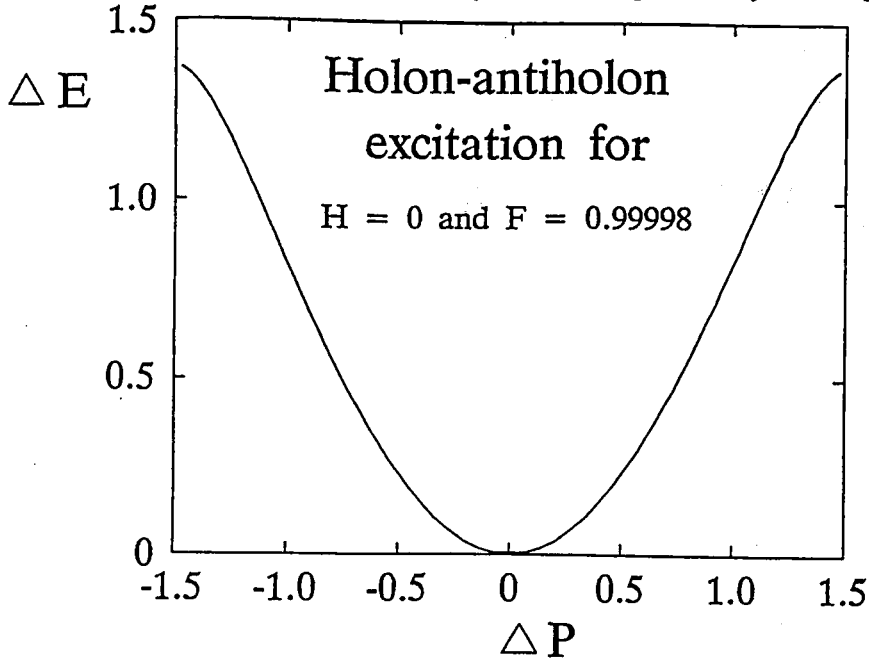


Fig.5: *Holon-antiholon excitation near half-filling*

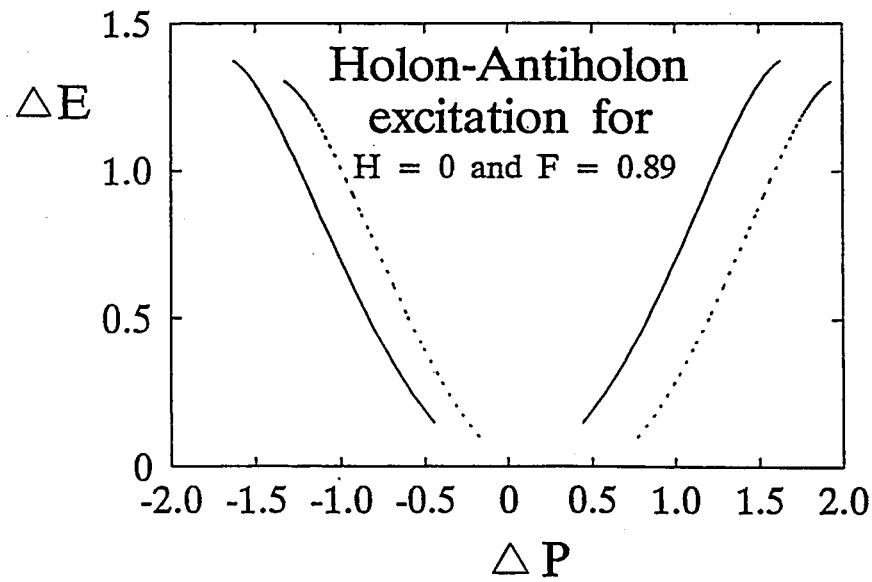


Fig.6: *Holon-antiholon excitation for* $F = 0.89$

Then, the energy and momentum excitations are written in terms of the shift-function $F_\rho(v|\gamma_p, \gamma_h)$ obtained from eqs.(5.8) and (5.9). In the figures below we show the dispersion relations of the holon-antiholon excitation calculated numerically using eqs.(5.11), (5.12), (5.8) and (5.9).

The fig. 5 corresponds to the case that the system in close to the half-filling ($F = 0.99998$ or $Q = 0.0001$). We observe a quadratic behavior between ΔE and ΔP.

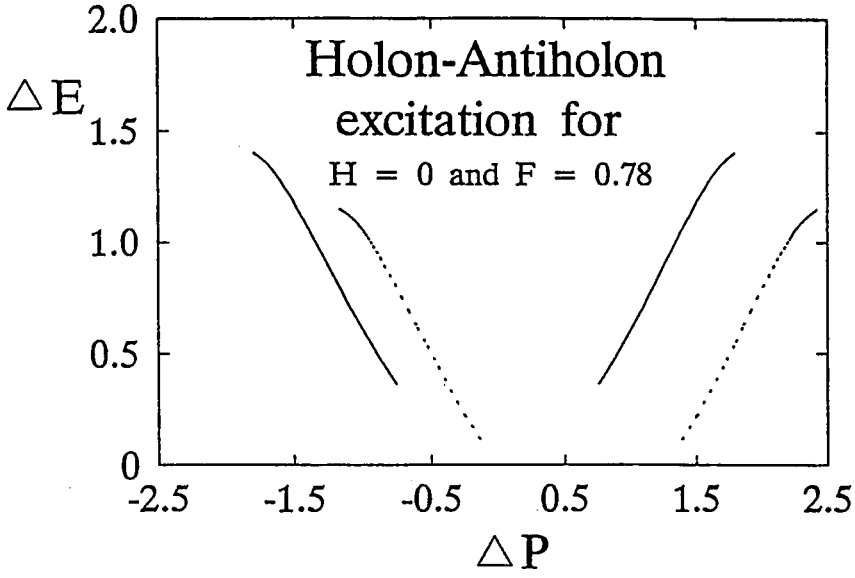


Fig.7: *Holon-antiholon excitation for* $F = 0.78$

By decreasing the filling we note that the dispersion law turns out to be linear. The different curves in the figs. 6 and 7 correspond to different choices of the antiholon position γ_h which is maintained fixed. The case $\gamma_h = Q$ is denoted by a dashed line while the case $\gamma_h = 0$ is represented by a solid line. Notice that when γ_h is very close to the Fermi level Q the excitation energy is nearly zero and it increases by moving the γ_h towards the origin.

We remark that the excitation energy ΔE (5.11) can be written in terms of the function ϵ in (3.2) through the relation

$$\Delta E(\gamma_p, \gamma_h) = \epsilon(\gamma_p) - \epsilon(\gamma_h) \tag{5.13}$$

The proof of this identity is analogous to the one for the Bose gas system [52]. Therefore, the function ϵ can be interpreted as the excitation energy measured from the Fermi level as seen in the chapter 2 by heuristic arguments.

5. SUMMARY

In this chapter we have studied the ground state and excitation spectrum of the one-dimensional supersymmetric t-J model. We have shown that using the Bethe ansatz formulation (eqs.(3.36)-(3.37) of chapter 2) the ground state involves only real roots. In addition, we have interpreted the excitation spectrum of the hamiltonian in terms of "quasi-particles" excitations as spinons and holons. We found that for small momenta the dispersion law for spinons is linear whereas the holon excitations at half-filling exhibits a quadratic behavior. The holon-antiholon excitation existing for non half-filling shows a linear dispersion behavior.

APPENDIX A : Proof of some useful relations

In this section we derive eqs.(2.22), (2.23) and (2.24) using the following properties

$$[n]const. = const., \tag{A.1}$$

$$[n][m] = [n+m], \tag{A.2}$$

$$[m]f_n = f_{m+n}; \quad f_n(v) = -\frac{n}{n^2+v^2}, \tag{A.3}$$

$$[j]A_{n,m} = \begin{cases} A_{n,m+j}, & m > n \\ A_{n+j,m}, & m \leq n \end{cases}, \tag{A.4}$$

$$[1](A_{n-1,m} + A_{n+1,m}) - ([0]+[2])A_{n,m} = -([0]+[2])\delta_{n,m}, \tag{A.5}$$

which can be obtained from the definition of $A_{n,m}$ (2.9) and $[n]$ (2.10). To exemplify the procedure we prove the relation (A.4). Suppose that $m > n$. Then, using (2.9) and (A.2) we find

$$\begin{aligned}
[j]A_{n,m} &= [m-n+j] + 2[m-n+j+2] + \cdots + 2[m+n+j-2] + 2[m+n+j] \\
&= [(m+j)-n] + 2[(m+j)-n+2] + \cdots + 2[(m+j)+n-2] + [m+j+n] \\
&= A_{m+j,n}
\end{aligned}$$

Analogously, for the case $m \leq n$ we have

$$[j]A_{n,m} = A_{n+j,m},$$

which completes the proof.

PROOF OF EQ.(2.22)

We begin by rewriting eq.(2.20) for the case $n = 1$

$$\ln(1+\varrho_1) = \frac{4f_1 + H}{T} + A_{1,1}\ln(1+\varrho_1^{-1}) + \sum_{m\geq 2} A_{1,m}\ln(1+\varrho_m^{-1}) - [1]\ln(1+\eta^{-1}) \tag{A.6}$$

where

$$\varrho_n \equiv \ln\frac{\rho_n^h}{\rho_n}, \quad \eta \equiv \ln\frac{\sigma^h}{\sigma} \tag{A.7}$$

From (2.9) we have that $A_{1,m} = [m-1]([0]+[2])$, which means that eq.(A.6) can be written in the form

$$\begin{aligned}
\ln(1+\varrho_1) =& \frac{4f_1+H}{T} + ([0]+[2])\left(\ln(1+\varrho_1) - \ln\varrho_1\right) + \\
&+([0]+[2])\sum_{m\geq 2}[m-1]\ln(1+\varrho_m^{-1}) - [1]\ln(1+\eta^{-1})
\end{aligned} \tag{A.8}$$

After some arrangements we have

$$\ln \varrho_1 = \frac{4f_1 + H}{T} + [2]\ln(1+\varrho_1^{-1}) + ([0]+[2])\sum_j [j]\ln(1+\varrho_{j+1}^{-1}) - [1]\ln(1+\eta^{-1}) \quad (A.9)$$

Multiplying both sides by T and using eqs.(A.7), (2.25) and (2.26) we obtain eq.(2.22).

Now we want to find a similar expression for $\varrho_n \geq 2$.

PROOF OF EQ.(2.23)

Combining eq.(2.20) together with (A.5) we find, after some manipulations

$$\begin{aligned} \ln \varrho_n =& [1]\ln(1+\varrho_{n-1}) + [2]\ln(1+\varrho_n^{-1}) \\ & \underbrace{-[2]\ln(1+\varrho_n) + [1]\ln(1+\varrho_{n+1})}_{AUX} \end{aligned} \quad (A.10)$$

Substituting (2.20) in the expression under brackets denoted by AUX and using the properties (A.1)-(A.4) we get

$$AUX = \sum_m \ln(1+\varrho_m^{-1})([1]A_{n+1,m} - [2]A_{n,m}) + \frac{H}{T} \quad (A.11)$$

By means of the eqs.(A.4) and (2.9) this relation can be simplified, resulting in

$$AUX = ([0]+[2]) \sum_{m=n+1} [m-n]\ln(1+\varrho_m^{-1}) + \frac{H}{T} \quad (A.12)$$

Substituting (A.12) in (A.10) we find

$$\begin{aligned} \ln \varrho_n =& [1]\ln(1+\varrho_{n-1}) + [2]\ln(1+\varrho_n^{-1}) \\ & ([0]+[2]) \sum_{m=n+1} [m-n]\ln(1+\varrho_m^{-1}) + \frac{H}{T} \end{aligned} \quad (A.13)$$

Multiplying both sites by T and using eqs.(A.7), (2.25) and (2.26) we obtain eq.(2.23).

The proof of eq.(2.24) is straightforward, we just have to use the fact that $\ln(1+\eta) - \ln(1+\eta^{-1}) = \ln \eta$ in the second relation of (2.20).

APPENDIX B : Fourier transforms and Wiener-Hopf method

In this appendix we present the definition of the Fourier Transforms which were extensively used in this chapter. We also review briefly the Wiener-Hopf method [57,61] employed to solve the integral equations (3.20), (3.21) and (3.31).

FOURIER TRANSFORMS

The function $\tilde{f}(x)$ is called the Fourier transform of $f(v)$, defined as

$$\tilde{f}(x) = \mathcal{F}(f(v)) \equiv \int_{-\infty}^{\infty} dv e^{-ivx} f(v) \tag{B.1}$$

and the inverse reads

$$f(v) = \mathcal{F}^{-1}\left(\tilde{f}(x)\right) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ivx}\tilde{f}(x). \tag{B.2}$$

A Fourier transform that appears frequently in this work is

$$\mathcal{F}\left(\frac{n}{n^2+v^2}\right) = \pi e^{-n|x|}, \quad n = 1, 2, \ldots \tag{B.3}$$

WIENER-HOPF METHOD

The Wiener-Hopf method is used to solve integral equations of the type

$$y(v) = \phi(v) + \int_0^{\infty} dw K(v-w)y(w). \tag{B.4}$$

By introducing the functions y_+ and y_- defined by

$$\begin{aligned} y_+(v) &= y(v), \quad v > 0 \\ y_-(v) &= y(v), \quad v < 0 \end{aligned} \tag{B.5}$$

we can Fourier transform (B.4), resulting in

$$(1 - \tilde{K}(x))\tilde{y}_+(x) + \tilde{y}_-(x) = \tilde{\phi}(x) \tag{B.6}$$

The key to the solution of this equation is to find a decomposition of the kernel K into factors $g_\pm$ that are analytic in the upper and lower complex x plane, respectively

$$1 - \tilde{K}(x) = \frac{1}{g_+(x)g_(x)}, \quad \lim_{x\to\infty} g_\pm(x) = 1. \tag{B.7}$$

Then, it can be proved that the Fourier transform of y_+ is given by [57,60]

$$\tilde{y}_+(x) = \frac{g_+(x)}{2\pi i}\int_{-\infty}^{\infty} dz \frac{\tilde{\phi}(z)}{z - x - i\eta} g_-(z). \tag{B.8}$$

Back transforming (B.8) and substituting into (B.4), we also obtain $y_-(v)$.

Chapter 4

The Supersymmetric t-J Model with Quantum Group Invariance

1. INTRODUCTION

The Yang-Baxter equation for the matrix of vertex weight S is the fundamental relation to construct integrable models. The solutions of the Yang-Baxter equation can be classified according to the dependence of the matrix S on the spectral parameter in rational, trigonometric and elliptic type. In the rational case (see for example eq.(2.5) in chapter 2), the Yang-Baxter algebra is associated to a simple Lie algebra. In the trigonometric case, it is a deformation of the Lie algebra called "quantum group" that underlies the Yang-Baxter algebra. In chapter 2 we have investigated the rational case for a graded 15-vertex model. Furthermore the connection between the Yang-Baxter algebra and the superalgebra $spl(2,1)$ was established. In the present chapter we analyze the corresponding trigonometric case and show how the concept of quantum supergroups appears in this context.

Although quantum groups are closely related to Yang-Baxter algebras, quantum group invariance holds only for integrable vertex models with special choices of boundary conditions. It has been observed [25,26] that the hamiltonian H_{XXZ} of the spin 1/2 anisotropic Heisenberg model with special boundary conditions and special endpoints terms

$$H_{XXZ} = -\frac{1}{2}\sum_{j=1}^{L-1}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \cos\gamma\sigma_j^z\sigma_{j+1}^z\right) + \frac{i}{2}\sin(\gamma)(\sigma_L^z - \sigma_1^z) \qquad (1.1)$$

is $SU_q(2)$ invariant ($q = e^{i\gamma}$), i.e., it commutes with the generators of the quantum group $SU_q(2)$. Although the hamiltonian (1.1) is not hermitean, its eigenvalues turn out to be real. This may be related to the fact that the hamiltonian (1.1) can be written in terms of the Temperly-Lieb operators [26]. Its spectrum was determined numerically by Alcaraz et al [25] using the coordinate Bethe ansatz. Furthermore, the diagonalization of the hamiltonian (1.1) through the quantum inverse scattering method was already considered by Cheridnik [27] and Sklyanin [28] and later by Destri and H. de Vega [41]. In these references the hamiltonian (1.1) was related to the 6-vertex model with anisotropy* and a generalization of the algebraic Bethe ansatz method was introduced in order to treat more general boundary conditions compared to the usual periodic ones. In this construction, in addition to the matrix S defining the vertex weights, two new matrices $K^{\pm}$ that take into account the boundary conditions are introduced. The explicit form of these matrices is determined by the requirement of integrability. In addition, in this approach the appropriate monodromy matrix is constructed from two horizontal lines of the two-dimensional classical vertex model. It is pointed out that the choice of special boundary conditions is essential to get "quantum group" invariance. However, quantum group invariance alone does not necessarily imply integrability [62], but this construction really provides a systematic way to get an integrable model with quantum group invariance. Other integrable models, as the Toda-chain [28,29], non-linear Schrödinger equation [28], the XY model [31], the XYZ model [28] and A_{n-1} vertex models [32] were also considered in connection with special boundary conditions. However, the Bethe ansatz equations were derived only for the XXZ chain [28]. Therefore, the problem of finding the spectrum of open chains has not been solved yet in its full extent and is presently a subject of increasing activity [63].

* In this context anisotropy means that the matrix S, which defines the vertex weights, depends on a new parameter (γ) in addition to the spectral parameter v.

In this chapter we introduce the quantum integrable supersymmetric t-J hamiltonian

$$\begin{aligned}\mathcal{H}^{(q)} = & -P\{\sum_{j=1}^{L-1}\sum_{\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma}+c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}P \\ & -2\sum_{j=1}^{L-1}\Big(S^x_jS^x_{j+1}+S^y_jS^y_{j+1}+\cos\gamma\ (S^z_jS^z_{j+1}-\frac{n_jn_{j+1}}{4})\Big)-\cos\gamma\sum_{j=1}^{L}n_j\,, \quad (1.2) \\ & +i\sin(\gamma)(n_1-n_L)-i\sin(\gamma)\sum_{j=1}^{L-1}(n_jS^z_{j+1}-S^z_jn_{j+1})\end{aligned}$$

which is $spl_q(2,1)$ invariant. We show that this hamiltonian can be related to the transfer matrix of a "graded" 15-vertex model with anisotropy adapted to special boundary conditions . Through a generalization of the Cheridnik and Sklyanin algebraic approach to the case of a graded three states vertex model we solve the eigenvalue problem of the transfer matrix in a recurrent way (nested Bethe ansatz method with two levels). The explicit form of the matrix S was obtained by Zhanget al [64],Kaufman [65]. For the matrices K^- and K^+ which define the boundary conditions we use the identity matrix and the Markov trace associated with the superalgebra $spl_q(2,1)$, respectively. This is motivated by the fact that in the theory of links the Markov trace is employed to build up invariant objects. In fact, we can verify that this choice leads to an integrable and invariant model. We remark that in references [28,41] the Markov trace does not appear explicitly. Nevertheless, the use of Markov traces allows for a straightforward generalization for more complex cases.

Obviously, the deformed t-J hamiltonian (1.2) is not hermitean, however it has real eigenvalues, as in the case of the quantum XXZ chain [25,26]. We remark that the choice of periodic boundary conditions for the "graded" 15-vertex model with anisotropy results in a hamiltonian with complex eigenvalues, as indicated by a direct diagonalization even for 3 sites. This is in contrast to the 6-vertex model, where the eigenvalues are also real for periodic boundary conditions. From the Yang-Baxter algebra of the model we construct in the limit $v \to \pm i\infty$ a deformation of the "graded" Lie algebra $spl(2,1)$. More precisely, the generators of the quantum supergroup $spl_q(2,1)$ emerge from these special limits of the spectral parameter in the monodromy matrix. We show explicitly that they commute with the transfer matrix.

The present chapter is organized as follows. In section 2 we introduce the $spl_q(2,1)$ vertex model and define some basic quantities, e.g., the vertex weights S and the matrices $K^{\pm}$ defining the boundary conditions, monodromy and transfer matrices. The relation between the transfer matrix of the vertex model and the hamiltonian of the supersymmetric t-J model is also discussed for the deformed case. The spectrum of the model is presented in sec.3 by means of the algebraic-nested Bethe-ansatz method adapted to special boundary conditions. In sec 4. we show that the integrable vertex model under consideration provides an explicit realization of the quantum supergroup $spl_q(2,1)$. A summary of our main results is presented in section 5.

2. THE VERTEX MODEL, YANG-BAXTER ALGEBRA AND SPECIAL BOUNDARY CONDITIONS

We consider an integrable generalization of the graded 15-vertex model (introduced in the chapter 2) with an anisotropy parameter γ and special boundary conditions [27,28,41]. It is a three states vertex model characterized by $\alpha = 1,2,3$, which can be bosonic (B) or fermionic (F). In what follows we will adopt the convention of chapter 2, i.e., 1 =B, 2 =B, 3 =F. We begin by introducing the matrix of vertex weights S, which in terms of a generic spectral parameter v reads [64,65]

$$S^{\gamma\delta}_{\alpha\beta}(v) \;=\; \gamma \xrightarrow[\beta]{v\ \ \delta} \alpha \;=\; \left(\begin{array}{ccc|ccc|ccc}
a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & b & 0 & c_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & 0 & 0 & 0 & c_- & 0 & 0 \\
\hline
0 & c_+ & 0 & b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b & 0 & c_- & 0 \\
\hline
0 & 0 & c_+ & 0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c_+ & 0 & b & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w
\end{array}\right) \qquad (2.1)$$

where the indices α , β , γ and δ run from 1 to 3 and

$$a = \sin(v+\gamma), \quad b = \sin v, \quad c_+ = e^{iv}\sin\gamma, \quad c_- = e^{-iv}\sin\gamma, \quad w = \sin(-v+\gamma). \quad (2.2)$$

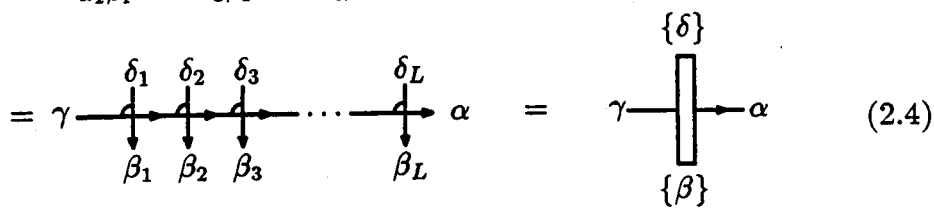
It provides a trigonometric solution of Yang-Baxter equation

$$S^{\alpha''\beta''}_{\alpha'\beta'}(v-v')S^{\alpha'\gamma''}_{\alpha\ \gamma'}(v)S^{\beta'\gamma'}_{\beta\ \gamma}(v') = S^{\beta''\gamma''}_{\beta'\gamma'}(v')S^{\alpha''\gamma'}_{\alpha'\ \gamma}(v)S^{\alpha'\beta'}_{\alpha\ \beta}(v-v'). \qquad (2.3)$$

In the limit $\gamma \to 0$ it reduces to the rational S matrix of chapter 2. This S matrix acts in the tensor product of two 3-dimensional auxiliary spaces $\mathbf{C}^3 \times \mathbf{C}^3$.

The standard row-to-row monodromy matrix for a $L \times L$ square lattice is defined as the matrix product over the S's

$$T^{\gamma\{\delta\}}_{\alpha\{\beta\}}(v) = S^{\gamma\ \delta_1}_{\alpha_2\beta_1}(v) S^{\alpha_2\delta_2}_{\alpha_3\beta_2}(v) S^{\alpha_3\delta_3}_{\alpha_4\beta_3}(v) \dots S^{\alpha_L\delta_L}_{\alpha\ \beta_L}(v)$$



(2.4)

This matrix T acts in the tensor product of an auxiliary space and a "quantum space" $\mathbf{C}^3 \times \mathbf{C}^{3L}$ and can be arranged as a 3×3 matrix of matrices acting in the "quantum space"

$$T^{\gamma}_{\alpha}(v) = \begin{pmatrix} A & B_2 & B_3 \\ C_2 & D^2_2 & D^2_3 \\ C_3 & D^3_2 & D^3_3 \end{pmatrix}. \tag{2.5}$$

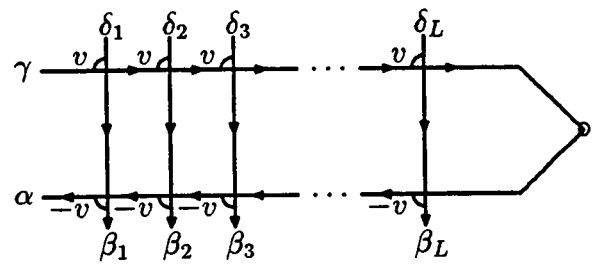
Furthermore, it also fulfills the Yang-Baxter relation

$$S^{\alpha''\beta''}_{\alpha'\beta'}(v-v') T^{\alpha'\{\gamma''\}}_{\alpha\ \{\gamma'\}}(v) T^{\beta'\{\gamma'\}}_{\beta\ \{\gamma\}}(v') = T^{\beta''\{\gamma''\}}_{\beta'\ \{\gamma'\}}(v') T^{\alpha''\{\gamma'\}}_{\alpha'\ \{\gamma\}}(v) S^{\alpha'\beta'}_{\alpha\ \beta}(v-v'), \tag{2.6}$$

as follows from eq.(2.3).

In order to construct the model with special boundary conditions let us introduce the "doubled" monodromy matrix $\mathcal{U}$ [28,41]

$$\mathcal{U}^{\gamma\{\delta\}}_{\alpha\{\beta\}}(v) = T^{\gamma\{\delta\}}_{\alpha'\{\beta'\}}(v) T^{-1\,\alpha'\{\beta'\}}_{\alpha\{\beta\}}(-v)$$



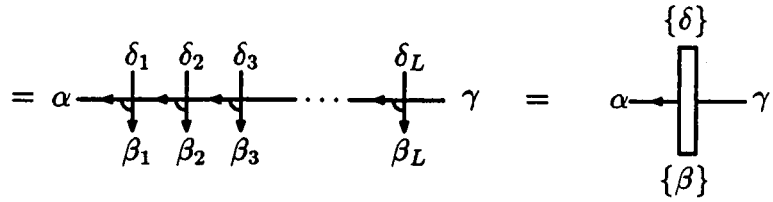
(2.7)

where the symbol o indicates that at this point the line denoted by v changes to $-v$ (in refs.[28,41] a matrix K^- is employed to indicate this change). T^{-1} is the inverse of T in the auxiliary and "quantum" spaces, i.e.,

$$T^{\gamma\ \{\delta\}}_{\alpha'\{\beta'\}}(v) T^{-1\,\alpha'\{\beta'\}}_{\alpha\ \{\beta\}}(v) = T^{-1\,\gamma\ \{\delta\}}_{\alpha'\{\beta'\}}(v) T^{\alpha'\{\beta'\}}_{\alpha\ \{\beta\}}(v) = \delta^{\gamma}_{\alpha}\delta^{\{\delta\}}_{\{\beta\}}, \tag{2.8}$$

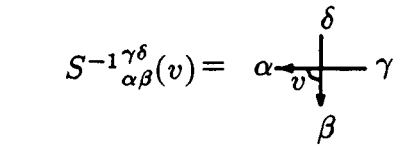
and can be constructed as the matrix product over the S^{-1} in the following way

$$T^{-1}{}_{\alpha\{\beta\}}^{\gamma\{\delta\}}(v) = S^{-1}{}_{\alpha\ \beta_1}^{\alpha_2\delta_1}(v) S^{-1}{}_{\alpha_2\beta_2}^{\alpha_3\delta_2}(v) S^{-1}{}_{\alpha_3\beta_3}^{\alpha_4\delta_3}(v) \ldots S^{-1}{}_{\alpha_L\beta_L}^{\gamma\ \delta_L}(v).$$



(2.9)

Here S^{-1} is the inverse of S



$$S^{-1}{}_{\alpha\beta}^{\gamma\delta}(v) = \quad , \tag{2.10}$$

$$S^{\gamma\ \delta}_{\alpha'\beta'}(v) S^{-1}{}_{\alpha\ \beta}^{\alpha'\beta'}(v) = S^{-1}{}^{\gamma\ \delta}_{\alpha'\beta'}(v) S^{\alpha'\beta'}_{\alpha\ \beta}(v) = \delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta}$$

and it can be obtained from the matrix S as

$$S^{-1}{}^{\gamma\ \delta}_{\alpha'\beta'}(v,\gamma) = -\frac{S^{\delta\ \gamma}_{\beta'\alpha'}(-v,\gamma)}{\sin(v+\gamma)\sin(v-\gamma)}. \tag{2.11}$$

The notation for the elements of S^{-1} and T^{-1} is the same as in (2.1) and (2.5) with a signal "~" to distinguish them from the elements of S and T. For example,

$$T^{-1}{}_{\alpha}^{\gamma}(v) = \begin{pmatrix} \tilde{A} & \tilde{B}_2 & \tilde{B}_3 \\ \tilde{C}_2 & \tilde{D}_2^2 & \tilde{D}_3^2 \\ \tilde{C}_3 & \tilde{D}_2^3 & \tilde{D}_3^3 \end{pmatrix}. \tag{2.12}$$

Analogously, the "doubled" monodromy matrix $\mathcal{U}$ can also be written as a 3×3 matrix

$$\mathcal{U}_{\alpha}^{\gamma}(v) = \begin{pmatrix} \mathcal{A} & \mathcal{B}_2 & \mathcal{B}_3 \\ \mathcal{C}_2 & \mathcal{D}_2^2 & \mathcal{D}_3^2 \\ \mathcal{C}_3 & \mathcal{D}_2^3 & \mathcal{D}_3^3 \end{pmatrix}. \tag{2.13}$$

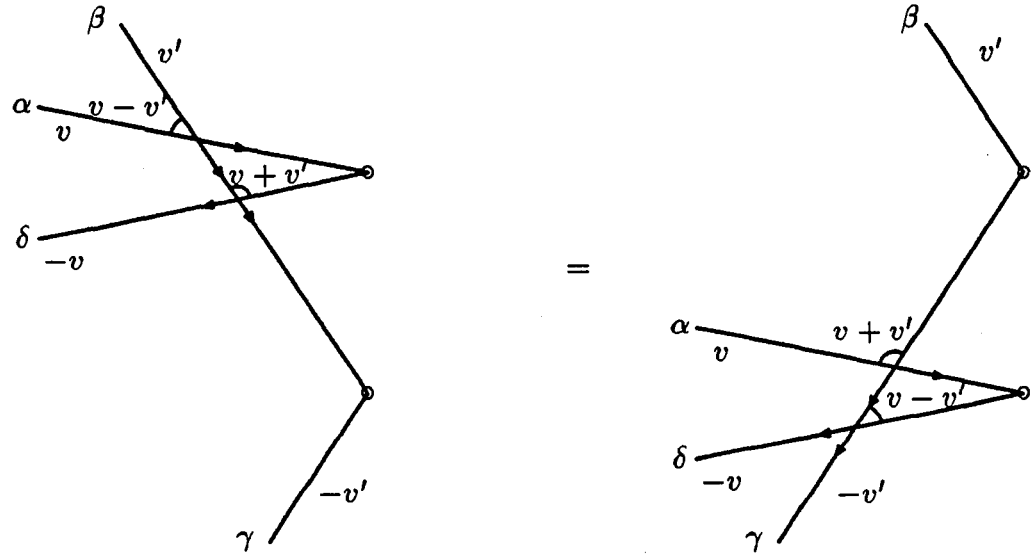
For later convenience, the "doubled" monodromy matrix is written as a product of matrices S and S^{-1}

$$\mathcal{U}_{\alpha\{\beta\}}^{\gamma\{\delta\}} = S^{\gamma\ \delta_1}_{\eta_2\beta_1'}(v) S^{-1}{}_{\alpha\ \beta_1}^{\nu_2\beta_1'}(-v) S^{\eta_2\delta_2}_{\eta_3\beta_2'}(v) S^{-1}{}_{\nu_2\beta_2}^{\nu_3\beta_2'}(-v) \ldots S^{\eta_L\delta_L}_{\alpha'\beta_L'}(v) S^{-1}{}_{\nu_L\beta_L}^{\alpha'\beta_L'}(-v), \tag{2.14}$$

which follows from (2.7), (2.4) and (2.9).

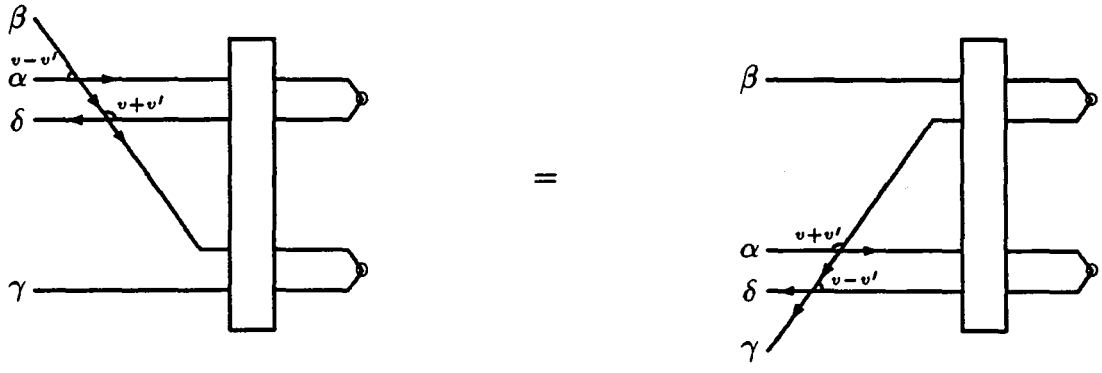
It can easily be shown that the matrix S satisfies the property

$$S^{\alpha\ \beta}_{\alpha'\beta'}(v-v')S^{\beta'\alpha'}_{\gamma\ \delta}(v+v') = S^{\alpha\ \beta}_{\alpha'\beta'}(v+v')S^{\beta'\alpha'}_{\gamma\ \delta}(v-v'). \tag{2.15}$$



Notice that in the language of refs.[28,41] we are working with $K^- = 1$. Then, using eqs.(2.6), (2.7) and (2.15) we can prove that the "doubled" monodromy matrix $\mathcal{U}$ fulfills a modified Yang-Baxter algebra of the following form (for simplicity, we omit the indices of the "quantum" space)

$$S^{\alpha\ \beta}_{\alpha'\beta'}(v-v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\beta'\delta'}_{\gamma'\delta}(v+v')\mathcal{U}^{\gamma'}_{\gamma}(v') = \mathcal{U}^{\beta}_{\beta'}(v')S^{\alpha\ \beta'}_{\alpha'\gamma'}(v+v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\gamma'\delta'}_{\gamma\ \delta}(v-v'). \tag{2.16}$$

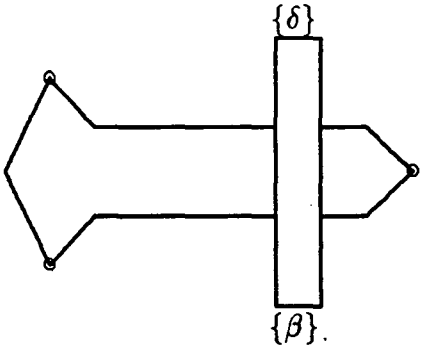


It is pointed out that we have a new structure of the Yang-Baxter algebra due to the presence of two vertex weights S at each side of the equation above. In fact, vertex models with special boundary conditions (S.B.C.) are more complex to handle in

comparison with vertex models with periodic boundary conditions (P.B.C.). Besides the fact that the monodromy matrix is constructed from two horizontal lines, in contrast to the usual one with only one line, the Yang-Baxter algebra has a more complicated structure.

The transfer matrix is defined as the Markov trace associated with the superalgebra $spl_q(2,1)$ of the "doubled" monodromy matrix in the auxiliary space

$$\tau^{\{\delta\}}_{\{\beta\}}(v) = \sum_{\alpha} K_{\alpha\alpha} \mathcal{U}^{\alpha\{\delta\}}_{\alpha\{\beta\}}(v), \tag{2.17}$$



where

$$K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix}. \tag{2.18}$$

Here q is the quantum group deformation parameter related to the anisotropy parameter γ by $q = e^{i\gamma}$, with q being root of unity. The Markov trace (2.18) is obtained from the projector on the trivial representation in $\mathbf{C}^3 \times \mathbf{C}^{3^*}$ [66]. In fact, Markov traces play a special role in the theory of links, where they are used to construct invariants [12,67]. Details about the derivation of this Markov trace are given in appendix A. We show in section 4 that this construction assures $spl_q(2,1)$ invariance of the transfer matrix of the model. In appendix A, we prove the integrability of the model, i.e., the transfer matrix (2.17) commutes for different spectral parameters

$$[\tau(v), \tau(v')] = 0. \tag{2.19}$$

Notice that in the construction of references [28,41] the matrix K depends on the spectral parameter v whereas in the Markov trace (2.18) v does not appear explicitly. This is

due to the fact that a symmetric form of the matrix S defining the vertex weights was employed in contrast to our case (see eq.(2.1)). We stress that the use of Markov traces is more useful, since it allows for a systematic generalization for more complicated cases (see, for example ref. [68], where a $SU_q(N)$ vertex model is studied in this context).

The transfer matrix (2.17) is related with a "q-deformed" version of the one-dimensional supersymmetric t-J model

$$\begin{aligned}\mathcal{H}^{(q)} = & - P\{\sum_{j=1}^{L-1}\sum_{\sigma}(c^{\dagger}_{j,\sigma}c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma}c_{j,\sigma})\}P \\ & - 2\sum_{j=1}^{L-1}\left(S^x_jS^x_{j+1} + S^y_jS^y_{j+1} + \cos\gamma\ (S^z_jS^z_{j+1} - \frac{n_jn_{j+1}}{4})\right) - \cos\gamma\sum_{j=1}^{L}n_j\,, \\ & + i\sin(\gamma)(n_1 - n_L) - i\sin(\gamma)\sum_{j=1}^{L-1}(n_jS^z_{j+1} - S^z_jn_{j+1})\end{aligned} \tag{2.20}$$

in the following way

$$\frac{\partial}{\partial v}\tau(v)|_{v=0} = \frac{-\sin\gamma}{4}\mathcal{H}^{(q)} + 2\cot\gamma \tag{2.21}$$

In (2.20) L is the number of sites of the quantum chain. The notation is the same as stated in chapter 2 for the supersymmetic t-J model.

Notice that the $spl_q(2,1)$ invariant hamiltonian (2.20) has special imaginary terms. In fact, this feature has also appeared in the $SU_q(2)$ invariant XXZ hamiltonian (see eq.(1.1)). However, in contrast to this case, the hamiltonian (2.20) has not only an imaginary boundary term $(n_1 - n_L)$ but also an imaginary term (last operator in (2.20)) that contributes, in principle, for all sites of the chain. In particular it gives a local contribution of $1(-1)$ when two first neighboring sites of the chain are occupied by $\downarrow\uparrow$ $(\uparrow\downarrow)$ and consequently it is relevant for configurations $\downarrow\uparrow$ $(\uparrow\downarrow)$ which are separated by holes, e.g., chains of the type $\uparrow\downarrow\ 0\ \uparrow\downarrow\ 0\ldots\ \uparrow\downarrow\ 0$. It is interesting to observe that this term breaks the spin parity of the model, which means that the hamiltonian (2.20) is not invariant by inverting all spins anymore. At half filling (one electron per site, i.e. $n_j = 1$ for $j = 1,\ldots,L$), this term reduces to the boundary term of the H_{XXZ} hamiltonian $(\frac{i}{2}\sin(\gamma)(\sigma^z_L - \sigma^z_1))$ and the boundary $(n_1 - n_L)$ cancels out, as expected. Although the deformed t-J hamiltonian is not hermitean, it possesses real eigenvalues. Indeed, we believe that a new definition of the scalar product (using the Markov trace) could restore the hermicity of the hamiltonian.

3. CONSTRUCTION OF BETHE EIGENVECTORS ADAPTED TO SPECIAL BOUNDARY CONDITIONS

In this section we will solve the eigenvalue problem of the transfer matrix (2.17)

$$\tau\Psi = \lambda\Psi, \tag{3.1}$$

following the algebraic nested Bethe ansatz method with two levels adapted to special boundary conditions (S.B.C.). This construction is based on the Yang-Baxter algebra of the "doubled" monodromy matrices

$$S^{\alpha}_{\alpha'}{}^{\beta}_{\beta'}(v-v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\beta'}_{\gamma'}{}^{\delta'}_{\delta}(v+v')\mathcal{U}^{\gamma'}_{\gamma}(v') = \mathcal{U}^{\beta}_{\beta'}(v')S^{\alpha}_{\alpha'}{}^{\beta'}_{\gamma'}(v+v')\mathcal{U}^{\alpha'}_{\delta'}(v)S^{\gamma'}_{\gamma}{}^{\delta'}_{\delta}(v-v'). \tag{3.2}$$

As already pointed out in the last section the "doubled" monodromy matrix $\mathcal{U}$ can be written as a 3×3 matrix

$$\left(\begin{array}{c|cc} \mathcal{A} & \mathcal{B}_2 & \mathcal{B}_3 \\ \hline \mathcal{C}_2 & \mathcal{D}_2^2 & \mathcal{D}_3^2 \\ \mathcal{C}_3 & \mathcal{D}_2^3 & \mathcal{D}_3^3 \end{array}\right). \tag{3.3}$$

and the transfer matrix is given by the Markov trace (2.18) of the "doubled" monodromy matrix $\mathcal{U}$ (see eq.(2.17)). According to the first level Bethe ansatz the eigenvector of the transfer matrix can be written as (in what follows we will omit the quantum space indices and write them only whenever necessary)

$$\Psi = \mathcal{B}_{\alpha_1}(v_1)\mathcal{B}_{\alpha_2}(v_2)\ldots\mathcal{B}_{\alpha_N}(v_N)\Phi\Psi^{\{\alpha\}}_{(1)}, \tag{3.4}$$

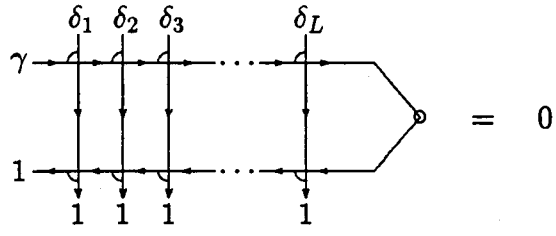
where the summations over the α_i $(i = 1,\ldots,N)$ are restricted to $\alpha_i = 2,\ 3$. The coefficients $\Psi^{\{\alpha\}}_{(1)}$ are determined by the second level Bethe ansatz and Φ is the first level "pseudovacuum" defined by the equation

$$\mathcal{C}_\gamma\Phi = 0 \qquad \text{for} \quad \gamma = 2,3, \tag{3.5}$$

whose solution is

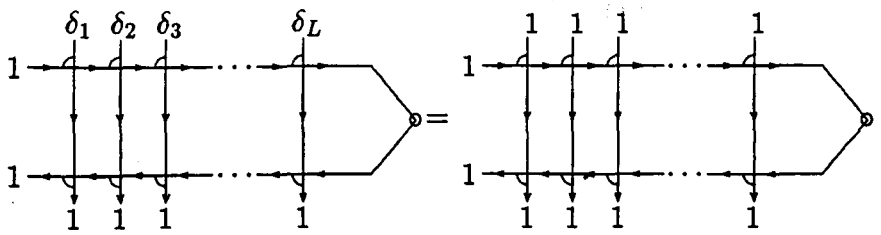
$$\Phi^{\{\beta\}} = \prod_{i=1}^{L}\delta_{\beta_i 1} = \begin{array}{ccccc} \beta_1 & \beta_2 & & & \beta_L \\ | & | & \cdots & & | \\ 1 & 1 & & & 1 \end{array}, \tag{3.6}$$

since



for all $\{\delta_i\}$. This pseudovacuum is an eigenvector of $\mathcal{A}$

$$\mathcal{A}(v)\Phi = a^L(v)\tilde{a}^L(-v)\Phi \tag{3.7}$$



and also of $\mathcal{D}_2^2$ and $\mathcal{D}_3^3$

$$\mathcal{D}_\alpha^\alpha(v)\Phi = -\Big(\frac{c_+(2v)}{a(2v)} - 1\Big)b^L(v)\tilde{b}^L(-v)\Phi + \frac{c_+(2v)}{a(2v)}\mathcal{A}(v)\Phi. \tag{3.8}$$

Here the graphic technic to get the eigenvalue is more cumbersome, since many different configurations are possible and we have to sum all of them. Nevertheless, an easy way to calculate $\mathcal{D}_\alpha^\alpha\Phi$ for $\alpha = 2, 3$ is to write $\mathcal{D}_\alpha^\alpha$ in terms of the elements of the monodromy T and its inverse T^{-1} (see eq.(2.7))

$$\mathcal{D}_\alpha^\alpha(v) = C_\alpha(v)\tilde{B}_\alpha(-v) + \sum_{\beta=2}^{3} D_\beta^\alpha(v)\tilde{D}_\alpha^\beta(-v) \tag{3.9}$$

and then commute C_α with $\tilde{B}_\alpha$ through the Yang-Baxter equation for the monodromy matrices T and T^{-1}, which follows straightforwardly from eq.(2.6). Finally, using the fact that $C_\alpha\Phi = \tilde{D}_2^3\Phi = \tilde{D}_3^2\Phi = 0$ we find eq.(3.8). We remark that in eqs. (3.8) and (3.9) no summation over α is assumed and the summations over the internal lines in eq.(3.7) can assume only the value 1, due to the special form of the matrices S and S^{-1}.

Following the general strategy of the algebraic Bethe ansatz to solve eq.(3.1) we apply the transfer matrix τ (2.17) to the eigenvector Ψ (3.4)

$$\tau(v)\Psi = \Big(\mathcal{A}(v) + \tau_D(v)\Big)\Psi, \tag{3.10}$$

where

$$\tau_D(v) = q^2\Big(\mathcal{D}_2^2 - \mathcal{D}_3^3\Big). \tag{3.11}$$

Using the Yang-Baxter relation (3.2) and the expressions for the matrices S (2.1) and $\mathcal{U}$ (2.13) we obtain the commutation rules between $\mathcal{A}(v)$, $\mathcal{D}_{\alpha'}^{\alpha}(v)$ and $\mathcal{B}_\alpha(v')$ (α

,$\alpha' = 2,3$). We do not write them here since we will not need them later. These relations simplify considerably if instead of using $\mathcal{D}^{\alpha}_{\alpha'}(v)$ we work with $\hat{\mathcal{D}}^{\alpha}_{\alpha'}(v)$ such that

$$\mathcal{D}^{\alpha}_{\alpha'}(v) = \chi(v)\frac{S^{\alpha\ \beta}_{\beta'\alpha'}(2v+\gamma)}{b(2v+\gamma)}\hat{\mathcal{D}}^{\beta'}_{\beta}(v) + \frac{c_+(2v)}{a(2v)}\delta^{\alpha}_{\alpha'}\mathcal{A}(v), \tag{3.12}$$

where $\chi(v)$ is given by

$$\chi(v) = q\frac{b(2v)a(2v+\gamma)}{a(2v)b(2v+\gamma)}. \tag{3.13}$$

More explicit we have

$$\begin{aligned}\mathcal{A}(v)\mathcal{B}_{\alpha}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{\alpha}(v')\mathcal{A}(v) - \frac{c_+(v'-v)b(2v')}{b(v'-v)a(2v')}\mathcal{B}_{\alpha}(v)\mathcal{A}(v') \\ &-\frac{c_-(v'+v)}{a(v'+v)}\chi(v')\mathcal{B}_{\alpha'}(v)\hat{\mathcal{D}}^{\beta}_{\beta'}(v')\frac{S^{\alpha'\beta'}_{\beta\ \alpha}(2v'+\gamma)}{b(2v'+\gamma)}\end{aligned} \tag{3.14}$$

and correspondingly for $\hat{\mathcal{D}}^{\beta}_{\beta'}$

$$\begin{aligned}\hat{\mathcal{D}}^{\beta}_{\beta'}(v)\mathcal{B}_{\alpha}(v') =& \frac{S^{t\beta''\alpha'}_{\beta'\ \alpha''}(v-v')}{b(v-v')}\frac{S^{t-1\beta\ \ \alpha''}_{\beta'''\alpha}(-v-v'-\gamma)}{\tilde{b}(-v-v'-\gamma)}\mathcal{B}_{\alpha'}(v')\hat{\mathcal{D}}^{\beta'''}_{\beta''}(v) \\ &+\frac{1}{\chi(v)}\frac{c_+(v+v')b(2v')}{a\ (v+v')a(2v')}\mathcal{B}_{\beta'}(v)\mathcal{A}(v')\delta^{\beta}_{\alpha} \\ &-\frac{\chi(v')}{\chi(v)}\frac{c_+(v-v')}{b(v-v')}\frac{S^{\beta\ \alpha''}_{\beta''\alpha}(2v'+\gamma)}{b(2v'+\gamma)}\mathcal{B}_{\beta'}(v)\hat{\mathcal{D}}^{\beta''}_{\alpha''}(v')\end{aligned} \tag{3.15}$$

All indices of the auxiliary space in eqs.(3.12), (3.14) and eq.(3.15) assume only the values 2 and 3 and the superscript "t" in eq.(3.15) means the transpose of the matrix S (2.1) ($S^{t\delta\gamma}_{\alpha\beta} = S^{\gamma\delta}_{\beta\alpha}$) and its inverse. We observe in these relations the presence of an additional term $\mathcal{B}_{\alpha}(v)\hat{\mathcal{D}}^{\beta}_{\beta'}(v')$ in (3.14) and $\mathcal{B}_{\beta'}(v)\mathcal{A}(v')$ in (3.15) in comparison with the corresponding relations using the Yang-Baxter algebra for the monodromy T (2.6). Nevertheless, the scheme of the algebraic Bethe ansatz method can be applied, as for the H_{XXZ} chain [28]. Notice that the action of the new operators $\hat{\mathcal{D}}^{\alpha'}_{\alpha}$ on the first level "pseudovacuum" Φ

$$\hat{\mathcal{D}}^{\alpha'}_{\alpha}(v)\Phi = \sigma^{\alpha'}_{\alpha}\frac{1}{q^2}\frac{b^2(2v+\gamma)}{a(2v+\gamma)w(2v+\gamma)}\Big(\frac{c_+(2v)}{a(2v)} - 1\Big)b^L(v)\tilde{b}^L(-v)\Phi, \tag{3.16}$$

yields a simpler expression than the corresponding one for $\mathcal{D}_\alpha^{\alpha'}(v)$. In fact, an examination of the action of $\mathcal{D}_\alpha^{\alpha'}(v)$ over the pseudovacuum Φ suggests how the operator $\hat{\mathcal{D}}_\alpha^{\alpha'}$ shall be defined. The matrix σ in eq.(3.16) is just a diagonal matrix given by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.17}$$

By successive applications of eq.(3.14) we commute $\mathcal{A}$ through all $\mathcal{B}_\alpha$ towards Φ. From the first term on the right hand side of (3.14) we get the "wanted" contributions

$$\mathcal{A}(v)\Psi = \lambda_A(v)\Psi + u.t.(\mathcal{A}), \tag{3.18}$$

where the coefficient λ_A is given by

$$\lambda_A = a^L(v)\tilde{a}^L(-v)\prod_{i=1}^{N}\frac{a(v_i - v)b(v_i + v)}{b(v_i - v)a(v_i + v)}. \tag{3.19}$$

The terms generated from the second and third terms in eqs. (3.14) are called "unwanted" since they can not originate a vector proportional to Ψ. Correspondingly, we get from the commutation relation (3.15) wanted and unwanted contributions

$$\begin{aligned}
\tau_D(v)\Psi =& q\chi(v)\sum_{\beta=2}^{3}\hat{\mathcal{D}}_\beta^\beta(v)\Psi = q\chi(v)\prod_{i=1}^{N}\Big(\frac{1}{b(v-v_i)}\frac{1}{\tilde{b}(-v-v_i-\gamma)}\Big) \\
&\times \mathcal{B}_{\alpha'_1}(v_1)\mathcal{B}_{\alpha'_2}(v_2)\ldots\mathcal{B}_{\alpha'_N}(v_N)\hat{\mathcal{D}}_{\beta''_N}^{\beta'''_N}(v)\Phi \\
&\times S^{t}{}_{\beta''_{N-1}j_N}^{\beta''_N\ \alpha'_N}(v-v_N)S^{t^{-1}}{}_{\beta'''_N\ \alpha_N}^{\beta'''_{N-1}j_N}(-v-v_N-\gamma)\ldots \\
&\times S^{t}{}_{\beta''_1 j_2}^{\beta''_2\alpha'_2}(v-v_2)S^{t^{-1}}{}_{\beta'''_2\alpha_2}^{\beta'''_1 j_2}(-v-v_2-\gamma) \\
&\times S^{t}{}_{\beta\ \ j_1}^{\beta''_1\alpha'_1}(v-v_1)S^{t^{-1}}{}_{\beta'''_1\alpha_1}^{\beta\ \ j_1}(-v-v_1-\gamma)\Psi_{(1)}^{\{\alpha\}} + u.t.(\mathcal{D}),
\end{aligned} \tag{3.20}$$

The product of S^t and $S^{t^{-1}}$ in the above expression can be recognized as a new "doubled" monodromy matrix $\mathcal{U}_{(1)}{}_{\beta'''_N\{\alpha\}}^{\beta''_N\{\alpha'\}}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})$, similar to $\mathcal{U}(v)$ (see eq.(2.14)). We denominate this matrix as the second level "doubled" monodromy matrix. Then, eq.(3.20) can be rewritten in the form

$$\begin{aligned}
\tau_D(v)\Psi =& q\chi(v)\hat{\mathcal{D}}_\beta^\beta\Phi = b^L(v)\tilde{b}^L(-v)\Big(\frac{c_+(2v)}{a(2v)}-1\Big)\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)} \\
&\times\prod_{i=1}^{N}\Big(\frac{1}{b(v-v_i)}\frac{1}{\tilde{b}(-v-v_i-\gamma)}\Big)\mathcal{B}_{\alpha'_1}(v_1)\mathcal{B}_{\alpha'_2}(v_2)\ldots\mathcal{B}_{\alpha'_N}(v_N)\cdot \\
&\times\hat{\mathcal{D}}_{\beta''_N}^{\beta'''_N}(v)\Phi\ \mathcal{U}_{(1)}{}_{\beta'''_N\{\alpha\}}^{\beta''_N\{\alpha'\}}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Psi_{(1)}^{\{\alpha\}} + u.t.(\mathcal{D}),
\end{aligned} \tag{3.21}$$

From eq.(3.16) we know how the operator $\hat{\mathcal{D}}^{\beta'''_N}_{\beta''_N}(v)$ acts on the first level pseudovacuum Φ. Thus, we have

$$\begin{aligned}\tau_D(v)\Psi =& q\chi(v)\hat{\mathcal{D}}^{\beta}_{\beta}\Phi = b^L(v)\tilde{b}^L(-v)\Big(\frac{c_+(2v)}{a(2v)}-1\Big)\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)}\\ &\times\prod_{i=1}^{N}\Big(\frac{1}{b(v-v_i)}\frac{1}{\tilde{b}(-v-v_i-\gamma)}\Big)\mathcal{B}_{\alpha'_1}(v_1)\mathcal{B}_{\alpha'_2}(v_2)\dots\mathcal{B}_{\alpha'_N}(v_N)\Phi \qquad (3.22)\\ &\times\tau_{(1)}{}^{\{\alpha'\}}_{\{\alpha\}}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Psi^{\{\alpha\}}_{(1)}+u.t.(\mathcal{D}),\end{aligned}$$

where $\tau_{(1)}$ is the second level transfer matrix

$$\tau_{(1)}(v,\{v_i\})=\sum_{\beta=2}^{3}\sigma^{\beta}_{\beta}\ \mathcal{U}_{(1)}{}^{\beta}_{\beta}(v,\{v_i\}), \qquad (3.23)$$

which is defined as the Markov trace associated to the superalgebra $SU_q(1,1)$ of the second level "doubled" monodromy matrix $\mathcal{U}_{(1)} = S^t(v-v_N)S^{t^{-1}}(v-v_N)\dots S^t(v-v_1)S^{t^{-1}}(v-v_1)$. It is pointed out that all indices range from 2 to 3, as in the internal block of the matrix $\mathcal{U}$ (3.3). This suggests the identification $\mathcal{A}_{(1)}\equiv\mathcal{U}_{(1)}{}^{2}_{2}$, $\mathcal{B}_{(1)}\equiv\mathcal{U}_{(1)}{}^{2}_{3}$, $\mathcal{C}_{(1)}\equiv\mathcal{U}_{(1)}{}^{3}_{2}$ and $\mathcal{D}_{(1)}\equiv\mathcal{U}_{(1)}{}^{3}_{3}$. Our task is to solve the eigenvalue problem

$$\tau_{(1)}(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Psi_{(1)}=\lambda_{(1)}\Psi_{(1)}, \qquad (3.24)$$

in order to get the eigenvalue $\lambda_{\mathcal{D}}$ in (3.22). The eigenvector $\Psi_{(1)}$ is defined by the second level Bethe ansatz

$$\Psi_{(1)}=\mathcal{B}_{(1)}(\nu_1+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\mathcal{B}_{(1)}(\nu_2+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\dots\mathcal{B}_{(1)}(\nu_M+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\})\Phi_{(1)}, \qquad (3.25)$$

where $\Phi_{(1)}$ is the second level "pseudovacuum", which takes the form $\Phi^{\{\alpha\}}_{(1)}=\prod_{i=1}^{N}\delta_{\alpha_i,2}$ and is annihilated by $\mathcal{C}_{(1)}$

$$\mathcal{C}_{(1)}(v,\{v_i\})\Phi_{(1)}=0. \qquad (3.26)$$

Indeed, as in the previous step, it is more convenient to work with the operator $\hat{\mathcal{D}}_{(1)}$ defined as

$$\hat{\mathcal{D}}_{(1)}(v,\{v_i\})=\mathcal{D}_{(1)}(v,\{v_i\})-\frac{c_-(2v)}{a(2v)}\mathcal{A}_{(1)}(v,\{v_i\}). \qquad (3.27)$$

Then the action of $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ on the pseudovaccum $\Phi_{(1)}$ is given by

$$\mathcal{A}_{(1)}(v,\{v_i\})\Phi_{(1)}=\prod_{i=1}^{N}a(v-v_i)\tilde{a}(-v-v_i)\Phi_{(1)}, \qquad (3.28)$$

$$\hat{\mathcal{D}}_{(1)}(v,\{v_i\})\Phi_{(1)} = \left(1-\frac{c_-(2v)}{a(2v)}\right)\prod_{i=1}^{N} b(v-v_i)\tilde{b}(-v-v_i)\Phi_{(1)}. \tag{3.29}$$

The commutation relations for $\mathcal{A}_{(1)}$, $\mathcal{D}_{(1)}$and $\mathcal{B}_{(1)}$ follow from the Yang-Baxter algebra (3.2), which is also valid for $\mathcal{U}_{(1)}(v,\{v_i\})$

$$\begin{aligned}\mathcal{A}_{(1)}(v)\mathcal{B}_{(1)}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{(1)}(v')\mathcal{A}_{(1)}(v) - \frac{c_-(v'-v)b(2v')}{b(v'-v)a(2v')}\mathcal{B}_{(1)}(v)\mathcal{A}_{(1)}(v') \\ & -\frac{c_+(v'+v)}{a(v'+v)}\mathcal{B}_{(1)}(v)\hat{\mathcal{D}}_{(1)}(v'),\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}\hat{\mathcal{D}}_{(1)}(v)\mathcal{B}_{(1)}(v') =& \frac{a(v'-v)b(v'+v)}{b(v'-v)a(v'+v)}\mathcal{B}_{(1)}(v')\hat{\mathcal{D}}_{(1)}(v) + \frac{c_-(v-v')b(2v)}{b(v-v')a(2v)}\mathcal{B}_{(1)}(v)\hat{\mathcal{D}}_{(1)}(v') \\ & -\frac{c_-(v'+v)b(2v)b(2v')}{a(v'+v)a(2v)a(2v')}\mathcal{B}_{(1)}(v)\mathcal{A}_{(1)}(v'),\end{aligned} \tag{3.31}$$

In this level we also observe the presence of the terms $\mathcal{B}_{(1)}(v)\ \hat{\mathcal{D}}_{(1)}(v')$ in (3.30) and $\mathcal{B}_{(1)}(v)\ \mathcal{A}_{(1)}(v')$ in (3.31). Then, proceeding along the same lines as in the calculation of (3.10) we obtain wanted and unwanted contributions when $\mathcal{A}_{(1)}$ and $\mathcal{D}_{(1)}$ commutes with $\mathcal{B}_{(1)}$ toward $\Phi_{(1)}$. The "unwanted" terms are not proportional to $\Psi_{(1)}$ and hence they must cancel in order to get an eigenvector of the second level transfer matrix $\tau_{(1)}$ (3.24).

$$\begin{aligned}\tau_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\Psi_{(1)} =& \left(\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\mathcal{A}_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\right. \\ & \left.-\hat{\mathcal{D}}_{(1)}\left(v+\frac{\gamma}{2},\{v_i+\frac{\gamma}{2}\}\right)\right)\Psi_{(1)} \\ =& \left(\lambda_{\mathcal{A}_{(1)}}+\lambda_{\mathcal{D}_{(1)}}\right)\Psi_{(1)} + u.t.(\mathcal{A}_{(1)}) + u.t.(\mathcal{D}_{(1)})\end{aligned}, \tag{3.32}$$

where $\lambda_{\mathcal{A}_{(1)}}$ and $\lambda_{\mathcal{D}_{(1)}}$ are given by

$$\lambda_{\mathcal{A}_{(1)}} = \left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\prod_{i=1}^{N} a(v-v_i)\tilde{a}(-v-v_i-\gamma)\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)} \tag{3.33}$$

$$\lambda_{\mathcal{D}_{(1)}} = -\left(1-\frac{c_-(2v+\gamma)}{a(2v+\gamma)}\right)\prod_{i=1}^{N} b(v-v_i)\tilde{b}(-v-v_i-\gamma)\prod_{j=1}^{M}\frac{a(\nu_j-v)b(\nu_j+v+\gamma)}{b(\nu_j-v)a(\nu_j+v+\gamma)} \tag{3.34}$$

Thus, substituting these results in (3.22) we find

$$\tau_{\mathcal{D}}(v)\Psi = \big(\lambda_{\mathcal{D}_I}(v) + \lambda_{\mathcal{D}_{II}}(v)\big)\Psi + u.t.(\mathcal{D}), \tag{3.35}$$

where the eigenvalues $\lambda_{\mathcal{D}_I}$ and $\lambda_{\mathcal{D}_{II}}$ read

$$\begin{aligned}\lambda_{\mathcal{D}_I}(v) = &\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)}\Big(\frac{c_+(2v)}{a(2v)} - 1\Big)\Big(1 - \frac{c_-(2v+\gamma)}{a(2v+\gamma)}\Big)\, b^L(v)\tilde{b}^L(-v) \\ &\times \prod_{i=1}^{N} \frac{a(v-v_i)\tilde{a}(-v-v_i-\gamma)}{b(v-v_i)\tilde{b}(-v-v_i-\gamma)} \prod_{j=1}^{M} \frac{a(\nu_j - v)b(\nu_j + v + \gamma)}{b(\nu_j - v)a(\nu_j + v + \gamma)}.\end{aligned} \tag{3.36}$$

$$\begin{aligned}\lambda_{\mathcal{D}_{II}}(v) = &-\frac{b(2v)b(2v+\gamma)}{a(2v)w(2v+\gamma)}\Big(\frac{c_+(2v)}{a(2v)} - 1\Big)\Big(1 - \frac{c_-(2v+\gamma)}{a(2v+\gamma)}\Big)\, b^L(v)\tilde{b}^L(-v) \\ &\times \prod_{j=1}^{M} \frac{a(\nu_j - v)b(\nu_j + v + \gamma)}{b(\nu_j - v)a(\nu_j + v + \gamma)}.\end{aligned} \tag{3.37}$$

Then, combining eqs.(3.18) and (3.35) we get the eigenvalue $\lambda(v)$ of the transfer matrix $\tau(v)$ if the "unwanted terms" u.t.($\mathcal{A}$) and u.t.($\mathcal{D}$) cancel out

$$\tau(v)\Psi = \lambda(v)\Psi, \tag{3.38}$$

with

$$\lambda(v) = \lambda_{\mathcal{A}}(v) + \lambda_{\mathcal{D}_I}(v) + \lambda_{\mathcal{D}_{II}}(v). \tag{3.39}$$

All "unwanted" terms vanish if the Bethe ansatz equations (BAE) hold

$$\begin{aligned}\Big(\frac{a(v_k)\tilde{a}(-v_k)}{b(v_k)\tilde{b}(-v_k)}\Big)^L \prod_{i\neq k}^{N} \frac{a(v_i - v_k)b(v_i + v_k)\tilde{b}(\ v_i + v_k + \gamma)}{a(v_k - v_i)a(v_i + v_k)\tilde{a}(-v_i - v_k - \gamma)} \\ \times \prod_{j=1}^{M} \frac{a(\nu_j + v_k + \gamma)b(\nu_j - v_k)}{b(\nu_j + v_k + \gamma)a(\nu_j - v_k)} = 1\ , \quad k = 1, \ldots, N,\end{aligned} \tag{3.40}$$

$$\prod_{i=1}^{N} \frac{a(\nu_l - v_i)\tilde{a}(-\nu_k - v_i - \gamma)}{b(\nu_l - v_i)\tilde{b}(-\nu_k - v_i - \gamma)} = 1\ , \quad l = 1, \ldots, M \tag{3.41}$$

In this case Ψ (3.4) is an eigenstate of the transfer matrix τ (2.17). These equations can be obtained by demanding that the eigenvalue $\lambda(v)$ (3.39) has no poles at $v = v_i$ and $v = \nu_j$. As a matter of fact, one can use this property (a necessary condition for the

eigenvalues) as a short-cut to derive the BAE. Using eq.(2.2) and making the change of variables $v \to \frac{iv\gamma}{2} - \frac{\gamma}{2}$, $\nu \to \frac{iv\gamma}{2} - \gamma$ the BAE can be written as

$$\left(\frac{\sinh\frac{\gamma}{2}(v_k - i)}{\sinh\frac{\gamma}{2}(v_k + i)}\right)^{2L} \prod_{i\neq k}^{N} \frac{\sinh\frac{\gamma}{2}(v_k - v_i + 2i)\sinh\frac{\gamma}{2}(v_k + v_i + 2i)}{\sinh\frac{\gamma}{2}(v_k - v_i - 2i)\sinh\frac{\gamma}{2}(v_k + v_i - 2i)} \times \prod_{j=1}^{M} \frac{\sinh\frac{\gamma}{2}(v_k - \nu_j - i)\sinh\frac{\gamma}{2}(v_k + \nu_j - i)}{\sinh\frac{\gamma}{2}(v_k - \nu_j + i)\sinh\frac{\gamma}{2}(v_k + \nu_j + i)} = 1 , \quad k = 1, \ldots, N, \tag{3.42}$$

$$\prod_{i=1}^{N} \frac{\sinh\frac{\gamma}{2}(\nu_l - v_i - i)\sinh\frac{\gamma}{2}(\nu_l + v_i - i)}{\sinh\frac{\gamma}{2}(\nu_l - v_i + i)\sinh\frac{\gamma}{2}(\nu_l + v_i + i)} = 1 , \quad l = 1, \ldots, M. \tag{3.43}$$

Therefore, we have reduced the eigenvalue problem of the transfer matrix τ to the solution of a system of coupled transcendental equations for the parameters v and ν.

A direct check of the above equations shows that for a given solution $\{v_k\}$ and $\{\nu_l\}$ of the BAE a negation of any single parameter $v_k \to -v_k$ or $\nu_l \to -\nu_l$ also leads to a solution of the equations (3.42),(3.43) and (3.43). This implies that is sufficient to consider only BAE roots with a positive real part. Other solutions can be generated just by inverting the sign of any single parameter v or ν. For the case that all real parts of the parameters v and ν are positive, we can define $2N$ variables λ_k and $2M$ variables u_l as

$$\lambda_k = v_k, \quad \lambda_{k+N} = -v_{N-k+1} \quad j = 1, \ldots N \tag{3.44}$$

and

$$u_l = \nu_l, \quad u_{l+M} = -\nu_{M-l+1} \quad l = 1, \ldots, M \tag{3.45}$$

Then eqs.(3.42) and (3.43) can be rewritten as

$$\left(\frac{\sinh\frac{\gamma}{2}(\lambda_k - i)}{\sinh\frac{\gamma}{2}(\lambda_k + i)}\right)^{2L} \prod_{i\neq k}^{2N} \frac{\sinh\frac{\gamma}{2}(\lambda_k - \lambda_i + 2i)}{\sinh\frac{\gamma}{2}(\lambda_k - \lambda_i - 2i)} \times \prod_{j=1}^{2M} \frac{\sinh\frac{\gamma}{2}(\lambda_k - u_j - i)}{\sinh\frac{\gamma}{2}(\lambda_k - u_j + i)} = 1 , \quad k = 1, \ldots, 2N, \tag{3.46}$$

$$\prod_{i=1}^{2N} \frac{\sinh\frac{\gamma}{2}(u_l - \lambda_i - i)}{\sinh\frac{\gamma}{2}(u_l - \lambda_i + i)} = 1 \ , \quad l = 1, \ldots, 2M \tag{3.47}$$

They are similar to the BAE for periodic boundary conditions on a $2L$ sites (see appendix B). Nevertheless, only particular solutions of the above equations, which must be symmetrically distributed with respect to the origin (see eqs.(3.44) and (3.45)) satisfy the original BAE (3.42)-(3.43).

4. QUANTUM GROUP STRUCTURE OF THE DEFORMED T-J MODEL

In this section we investigate the quantum group structure of the deformed t-J model (2.20). We show that for special limit values of the spectral parameter v the Yang-Baxter algebra leads to a deformation of the "graded" Lie algebra $spl(2,1)$. Furthermore, we prove that the transfer matrix τ (or, equivalently, the deformed t-J hamiltonian) is $spl_q(2,1)$ invariant.

In what follows we use $x = e^{iv}$ and $q = e^{i\gamma}$. First it is convenient to write the matrix S (2.1) as

$$S(x) = xS_+ - x^{-1}S_- \quad , \tag{4.1}$$

then S_+ (S_-) corresponds to the leading term in the limit of the matrix $S(x)$ for $x \to \infty(0)$. They can be written as

$$S_+ = \begin{pmatrix} q^{w_1} & 0 & 0 \\ (q-q^{-1})q^{-\frac{w_3}{2}} f_1 & q^{w_2} & 0 \\ (q-q^{-1})[f_1, f_2] & (q-q^{-1})q^{-\frac{w_1}{2}-w_3}\sigma f_2 & \sigma q^{-w_3} \end{pmatrix}, \tag{4.2}$$

$$S_- = \begin{pmatrix} q^{-w_1} & -(q-q^{-1})e_1 q^{\frac{w_3}{2}} & -(q-q^{-1})[e_2, e_1] \\ 0 & q^{-w_2} & -(q-q^{-1})e_2 q^{\frac{w_1}{2}+w_3} \\ 0 & 0 & \sigma q^{w_3} \end{pmatrix}, \tag{4.3}$$

where w_i,f_i, e_i, $i = 1,2$ are the generators of the spl(2,1) superalgebra in the "graded" Cartan-Chevalley basis and the matrix σ was already defined in (3.17). Further details may be found in the appendix C, where we recall the main properties of the superalgebra spl(2,1) and also the fundamental representation of the generators w_i,f_i, e_i. In the limit $x \to \infty(0)$, the leading terms T_+, (T_-) of the monodromy matrix T (2.4) are proportional to

$$T_+ \propto \begin{pmatrix} q^{-\frac{L}{2}} q^{W_1} & 0 & 0 \\ \alpha_- q^{-\frac{W_3}{2}} F_1 & q^{-\frac{L}{2}} q^{W_2} & 0 \\ q^{\frac{L}{2}} T_{+1}^{\ 3} & \alpha_- q^{-\frac{W_1}{2}-W_3} \tilde{\sigma} F_2 & q^{-\frac{L}{2}} \tilde{\sigma} q^{-W_3} \end{pmatrix}, \tag{4.4}$$

$$T_- \propto \begin{pmatrix} q^{\frac{L}{2}} q^{-W_1} & -\alpha_+ E_1 q^{\frac{W_3}{2}} & q^{-\frac{L}{2}} T_{-3}^{\ 1} \\ 0 & q^{\frac{L}{2}} q^{-W_2} & -\alpha_+ E_2 q^{\frac{W_1}{2}+W_3} \\ 0 & 0 & q^{\frac{L}{2}} \tilde{\sigma} q^{W_3} \end{pmatrix}, \tag{4.5}$$

with $\alpha_\pm = q^{\pm\frac{1}{2}}(q - q^{-1})$ and

$$\begin{aligned} q^{\pm W_i} &= q^{\pm w_i} \otimes q^{\pm w_i} \otimes \ldots \otimes q^{\pm w_i}, \quad i = 1,2,3 \\ \tilde{\sigma} &= \sigma \otimes \sigma \otimes \ldots \otimes \sigma \end{aligned} \tag{4.6}$$

$$\begin{aligned} F_1 &= \sum_{j=1}^{L} q^{-\frac{h_1}{2}} \otimes \ldots \otimes q^{-\frac{h_1}{2}} \otimes \overset{j_{th}}{f_1} \otimes q^{\frac{h_1}{2}} \otimes \ldots \otimes q^{\frac{h_1}{2}} \\ F_2 &= \sum_{j=1}^{L} q^{-\frac{h_2}{2}} \otimes \ldots \otimes q^{-\frac{h_2}{2}} \otimes \overset{j_{th}}{f_2} \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \otimes \ldots \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \\ E_1 &= \sum_{j=1}^{L} q^{-\frac{h_1}{2}} \otimes \ldots \otimes q^{-\frac{h_1}{2}} \otimes \overset{j_{th}}{e_1} \otimes q^{\frac{h_1}{2}} \otimes \ldots \otimes q^{\frac{h_1}{2}} \\ E_2 &= \sum_{j=1}^{L} q^{-\frac{h_2}{2}} \otimes \ldots \otimes q^{-\frac{h_2}{2}} \otimes \overset{j_{th}}{e_2} \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \otimes \ldots \otimes \left(\sigma q^{\frac{h_2}{2}}\right) \end{aligned} \tag{4.7}$$

where $h_1 = w_1 - w_2$ and $h_2 = w_2 + w_3$ (see appendix C). In the last four expressions we have a kind of "q-sum" of one-site operators extended to all sites L. For $L = 2$ these formulae define a co-product : $\Delta^{(2)}(f_1) = F_1$, etc. The matrix σ takes into account that F_2 and E_2 are odd operators. In the literature we also find the definition of the coproduct without σ, but then we shall use the graded-tensor product law instead of usual tensor product law [69]. Both methods are equivalent. By taking appropriate limits ($x \to \infty, 0$ or $v \to \pm i\infty$) in the Yang-Baxter relation for the monodromy T (2.6) we get

$$S_{+\alpha'\beta'}^{\alpha''\beta''} T_{+\alpha}^{\alpha'} T_{+\beta}^{\beta'} = T_{+\beta'}^{\beta''} T_{+\alpha'}^{\alpha''} S_{+\alpha\ \beta}^{\alpha'\beta'}, \tag{4.8}$$

$$S_{+\alpha'\beta'}^{\alpha''\beta''} T_{+\alpha}^{\alpha'} T_{-\beta}^{\beta'} = T_{-\beta'}^{\beta''} T_{+\alpha'}^{\alpha''} S_{+\alpha\ \beta}^{\alpha'\beta'}, \tag{4.9}$$

$$S_{-\alpha'\beta'}^{\alpha''\beta''} T_{-\alpha}^{\alpha'} T_{-\beta}^{\beta'} = T_{-\beta'}^{\beta''} T_{-\alpha'}^{\alpha''} S_{-\alpha\ \beta}^{\alpha'\beta'}, \tag{4.10}$$

Using the definitions of S_+ (4.2),S_- (4.3),T_+ (4.4) and T_- (4.5) together with the above relations for special values of the free indices (α'',β'', α and β) we obtain the following q-(anti)commutation rules and cubic q-Serre relations (with $H_1 = W_1 - W_2$ and $H_2 = W_2 + W_3$)

$$\begin{aligned} q^{H_1}q^{H_2} &= q^{H_2}q^{H_1}, \\ q^{H_i}F_j q^{-H_i} &= q^{a_{ij}}F_j, \\ q^{H_i}E_j q^{-H_i} &= q^{-a_{ij}}E_j, \end{aligned} \tag{4.11}$$

$$\begin{aligned} [F_1, E_1] &= \frac{q^{H_1} - q^{-H_1}}{q - q^{-1}}, \quad [F_1, E_2] = 0, \\ [F_2, E_2]_+ &= \frac{q^{H_2} - q^{-H_2}}{q - q^{-1}}, \quad [F_2, E_1] = 0, \\ [E_2, E_2]_+ &= [F_2, F_2]_+ = 0, \end{aligned} \tag{4.12}$$

$$\begin{aligned} \left(F_1\right)^2 F_2 - (q + q^{-1})F_1 F_2 F_1 + F_2\left(F_1\right)^2 &= 0, \\ \left(E_1\right)^2 E_2 - (q + q^{-1})E_1 E_2 E_1 + E_2\left(E_1\right)^2 &= 0, \\ qF_1\left(F_2\right)^2 - q^{-1}\left(F_2\right)^2 F_1 &= 0, \\ qE_1\left(E_2\right)^2 - q^{-1}\left(E_2\right)^2 E_1 &= 0. \end{aligned} \tag{4.13}$$

Here a_{ij} is an element of the "graded" Cartan matrix given by $a_{11} = 2$, $a_{ij} = -1 (i = j \pm 1)$, $a_{ij} = 0$ (otherwise), with $i, j = 1, 2$. In the appendix D we show further details about the calculation of the above expressions. These relations define a deformation of the $spl(2,1)$ superalgebra since in the isotropic limit $\gamma \to 0$ the usual $spl(2,1)$ (anti-) commutators and cubic Serre relations is recovered. It is called the $spl_q(2,1)$ quantum group. Note that this deformed structure is x independent and depends only on the anisotropic parameter q and that the complicated commutation rules (4.11) – (4.13) are equivalent to the beautiful relations (4.8) – (4.10) which follow from the Yang-Baxter equation (2.6).

Let us now consider the "doubled" monodromy matrix $\mathcal{U}$. From eqs.(2.7) and (2.13) we can write the elements of $\mathcal{U}$ in terms of the row-to-row monodromy T (2.5) and its inverse T^{-1} (2.12)

$$\begin{aligned}
\mathcal{A}(x) &= A(x)\tilde{A}(1/x) + \sum_{\alpha=2}^{3} B_\alpha(x)\tilde{C}_\alpha(1/x), \\
\mathcal{B}_\alpha(x) &= A(x)\tilde{B}_\alpha(1/x) + \sum_{\beta=2}^{3} B_\beta(x)\tilde{D}^\beta_\alpha(1/x), \\
\mathcal{C}_\alpha(x) &= C_\alpha(x)\tilde{A}(1/x) + \sum_{\beta=2}^{3} D^\alpha_\beta(x)\tilde{C}_\beta(1/x), \\
\mathcal{D}^\alpha_\beta(x) &= C_\alpha(x)\tilde{B}_\beta(1/x) + \sum_{\gamma=2}^{3} D^\alpha_\gamma(x)\tilde{D}^\gamma_\beta(1/x),
\end{aligned} \tag{4.14}$$

where the indices α,β assume the values 2 or 3. Then, taking the limit $x \to \infty, 0$ of the "doubled" monodromy matrix $\mathcal{U}$ we find that the leading terms also yield the generators of $spl_q(2,1)$. We list below just the relevant ones for next calculations

$$\begin{aligned}
\mathcal{A}(x \to \infty) &\sim q^{-L} q^{2W_1}, \\
\mathcal{D}^3_3(x \to 0) &\sim q^{L} q^{2W_3}, \\
\mathcal{B}_2(x \to \infty) &\sim \alpha_- q^{-\frac{L}{2}} q^{W_1} E_1 q^{-\frac{1}{2}W_3}, \\
\mathcal{C}_2(x \to \infty) &\sim \alpha_- q^{-\frac{L}{2}} q^{-\frac{1}{2}W_3} F_1 q^{W_1}, \\
\mathcal{D}^2_3(x \to 0) &\sim -\alpha_+ q^{\frac{L}{2}} E_2 q^{\frac{1}{2}W_1 + 2W_3} \tilde{\sigma}, \\
\mathcal{D}^3_2(x \to 0) &\sim -\alpha_+ q^{\frac{L}{2}} \tilde{\sigma} q^{\frac{1}{2}W_1 + 2W_3} F_2.
\end{aligned} \tag{4.15}$$

This means the $spl_q(2,1)$ generators can also be obtained from $\mathcal{U}$ and using the Yang-Baxter relation for the "doubled" monodromy $\mathcal{U}$ we also get the q-(anti)commutation and cubic q-Serre relations (4.11) – (4.13). It should be stressed that the transfer matrix constructed from T with periodic boundary conditions is not "quantum group" invariant since it does not commute with the $spl_q(2,1)$ generators. In fact, this feature also appears in the six vertex model with anisotropy [41]. In addition, the eigenvalues of this transfer matrix are not real. In order to get quantum group invariance, we shall work

with the "doubled" monodromy matrix $\mathcal{U}$ (associated with special boundary conditions). This can be shown as follows. Letting $x' \to \infty, 0$ in eq.(3.2) we can evaluate the relevant commutation rules between $\mathcal{A}(x)$,$\mathcal{D}_2^2(x)$ and $\mathcal{D}_3^3(x)$ and the $spl_q(2,1)$ generators. After some calculations we obtain

$$[\mathcal{A}(x), E_1] = -q^2[\mathcal{D}_2^2(x), E_1] = -q^{-\frac{1}{2}-\frac{L}{2}} q^{-W_1} \mathcal{B}_2(x) q^{2W_1} q^{\frac{1}{2}W_3}, \tag{4.16}$$

$$[\mathcal{D}_3^3(x), E_1] = 0,$$

$$[\mathcal{D}_2^2(x), E_2] = [\mathcal{D}_3^3(x), E_2] = -q^{\frac{1}{2}+\frac{L}{2}} \mathcal{D}_3^2(x) q^{-\frac{1}{2}W_1}, \tag{4.17}$$

$$[\mathcal{A}(x), E_2] = 0,$$

$$[\mathcal{A}(x), F_1] = -q^2[\mathcal{D}_2^2(x), F_1] = q^{\frac{3}{2}-\frac{L}{2}} q^{\frac{1}{2}W_3} \mathcal{C}_2(x) q^{W_1}, \tag{4.18}$$

$$[\mathcal{D}_3^3(x), F_1] = 0,$$

$$[\mathcal{D}_2^2(x), F_2] = [\mathcal{D}_3^3(x), F_2] = -q^{-\frac{3}{2}+\frac{L}{2}} q^{-\frac{1}{2}W_1 - 2W_3} \tilde{\sigma} \mathcal{D}_2^3(x) q^{2W_3}, \tag{4.19}$$

$$[\mathcal{A}(x), F_2] = 0,$$

$$[\mathcal{A}(x), q^{W_i}] = [\mathcal{D}_2^2(x), q^{W_i}] = [\mathcal{D}_3^3(x), q^{W_i}] = 0, \quad i = 1,2,3. \tag{4.20}$$

In the appendix E we present explicitly the calculation of one of these relations, namely the commutation between $\mathcal{A}(x)$ and F_1. Using these commutation relations and the definition of the matrix $\tau = tr(K\mathcal{U})$ in eq.(2.17) we easily obtain

$$\begin{aligned} &[\tau(x), E_1] = [\tau(x), E_2] = 0 \\ &[\tau(x), F_1] = [\tau(x), F_2] = 0 \\ &[\tau(x), q^{H_i}] = 0, \quad i = 1,2. \end{aligned} \tag{4.21}$$

Therefore, the transfer matrix τ is $spl_q(2,1)$ invariant and consequently the eigenstates of the deformed t-J hamiltonian (2.20) can be classified by the $spl_q(2,1)$ superalgebra.

The question of the completeness of the Bethe states for the deformed t-J model was not considered so far. In fact, this does not to seem to be an easy task, since it was not proved yet even for the Heisenberg H_{XXZ} model. There is however some hope that the quantum group invariance could bring some insight on the problem of the completeness.

5. SUMMARY

In this chapter we have introduced a new integrable vertex model, namely, the "graded" 15-vertex model with anisotropy. Through a generalization of the Cherdnik-Sklyanin construction for the case of a vertex model with three states, which can be bosonic or fermionic, we derived the Bethe ansatz equations of the model. Moreover, we showed that this vertex model can be related to a deformed one-dimensional supersymmetric t-J model.

The underlying quantum group structure was also investigated. We found that a deformation of the "graded" Lie algebra $spl(2,1)$ follows from the Yang Baxter algebra in the limit $v \to \pm i\infty$. Finally, we have proved that this model is $spl_q(2,1)$ quantum group invariant.

APPENDIX A : Integrability

In this appendix we prove that the transfer matrix $\tau(v)$ (2.17) commutes for different spectral parameters. Let us introduce the necessary notation for the S-matrix and Clebsch-Gordan coefficients.

$$S^{\gamma\delta}_{\alpha\beta}(v) = \quad = \left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c_- & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c_- & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & c_+ & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c_- & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & c_+ & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_+ & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{array}\right) \qquad (A.1)$$

$$\bar{S}^{\bar{\gamma}\bar{\delta}}_{\bar{\alpha}\bar{\beta}}(v) = \quad = \left(\begin{array}{ccc|ccc|ccc} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & 0 & c_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & c_- & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & c_- & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b & 0 & c_- & 0 \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & c_+ & 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_+ & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{array}\right) \qquad (A.2)$$

$$K^{\vee}_{\bar{\alpha}\alpha} = \quad = \begin{pmatrix} 0 & q & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -q \end{pmatrix} \qquad (A.3)$$

$$K^{\wedge}_{\bar{\alpha}\alpha} = \quad = \begin{pmatrix} 0 & q & 0 \\ -1 & 0 & 0 \\ 0 & 0 & q \end{pmatrix} \qquad (A.4)$$

Notice that the Markov trace associated with the superalgebra $spl_q(2,1)$ (2.18) can be obtained from the Clebsh-Gordan coeficientes (A.3) and (A.4) as follows

$$K_{\alpha\beta} = \quad = \sum_{\bar{\gamma}} K^{\wedge}_{\bar{\gamma}\alpha} K^{\vee}_{\bar{\gamma}\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & -q^2 \end{pmatrix} \qquad (A.5)$$

In order to prove the intebrabilty of the model we use the modified Yang-Baxter algebra (2.16), the relation (2.15) and the following properties

$$\bar{S}^{\bar{\beta}'\bar{\beta}}_{\bar{\alpha}\bar{\alpha}'}(\tfrac{x}{q}) K^{\wedge}_{\bar{\alpha}'\alpha'} K^{\vee}_{\bar{\beta}'\beta'} S^{\alpha'\alpha}_{\beta\beta'}(\tfrac{1}{x}) = c_1(x) K^{\vee}_{\bar{\alpha}\alpha} K^{\wedge}_{\bar{\beta}\beta}$$

$= c_1(x)$ (A.6)

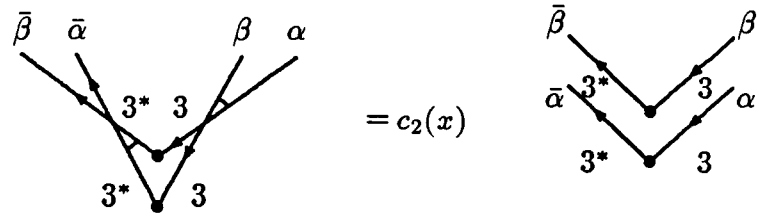
where $c_1(x) = q + q^{-1} - \frac{q}{x^2} - \frac{x^2}{q}$

$$\bar{S}^{\bar{\alpha}\bar{\beta}}_{\bar{\beta}'\bar{\alpha}'}(x) K^{\wedge}_{\bar{\beta}'\beta'} K^{\wedge}_{\bar{\alpha}'\alpha'} S^{\alpha'\beta'}_{\beta\alpha}(\tfrac{1}{x}) = c_2(x) K^{\wedge}_{\bar{\alpha}\alpha} K^{\wedge}_{\bar{\beta}\beta}$$

$= c_2(x)$ (A.7)

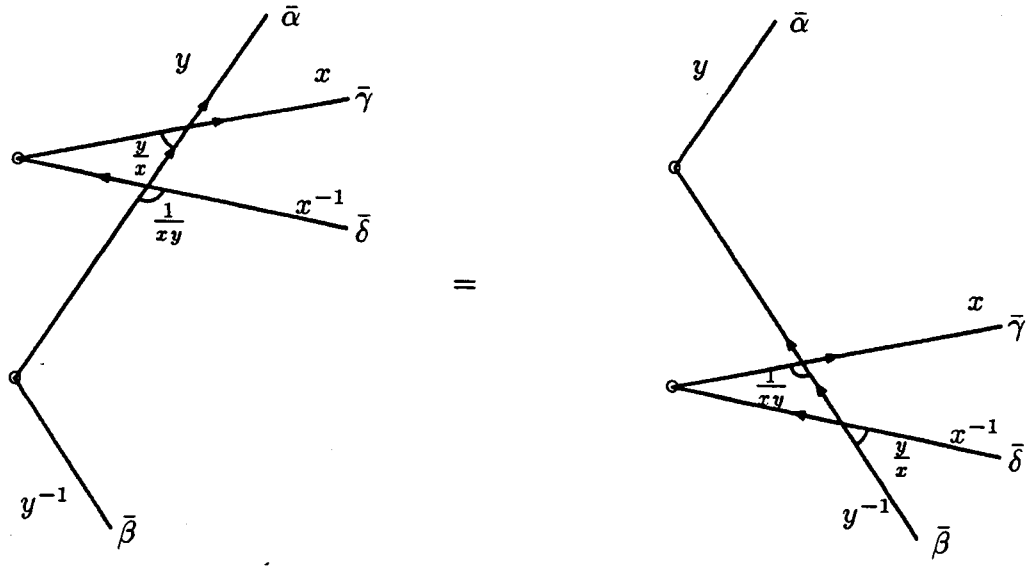
where $c_2(x) = q^2 + q^{-2} - x^2 - x^{-2}$

$$\bar{S}^{\bar{\alpha}'\bar{\beta}'}_{\bar{\beta}\bar{\alpha}}(x)K^{\vee}_{\bar{\beta}'\beta'}K^{\vee}_{\bar{\alpha}'\alpha'}S^{\alpha\beta}_{\beta'\alpha'}(\tfrac{1}{x}) = c_2(x)K^{\vee}_{\bar{\alpha}\alpha}K^{\vee}_{\bar{\beta}\beta}$$



(A.8)

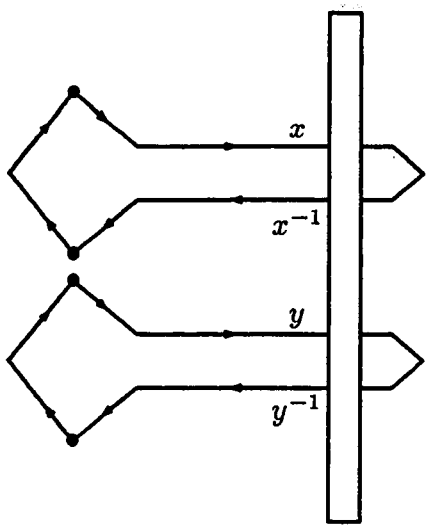
$$\bar{S}^{\bar{\beta}'\bar{\alpha}'}_{\bar{\alpha}\bar{\gamma}}(\tfrac{y}{x})\bar{S}^{\bar{\delta}\bar{\beta}}_{\bar{\alpha}'\bar{\beta}'}(\tfrac{1}{xy}) = \bar{S}^{\bar{\alpha}'\bar{\beta}'}_{\bar{\alpha}\bar{\gamma}}(\tfrac{1}{xy})\bar{S}^{\bar{\delta}\bar{\beta}}_{\bar{\beta}'\bar{\alpha}'}(\tfrac{y}{x})$$



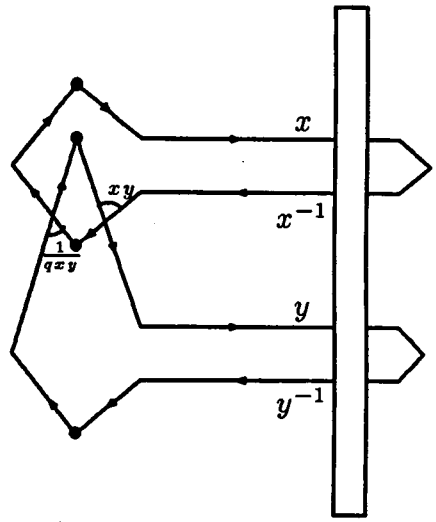
(A.9)

Proof :

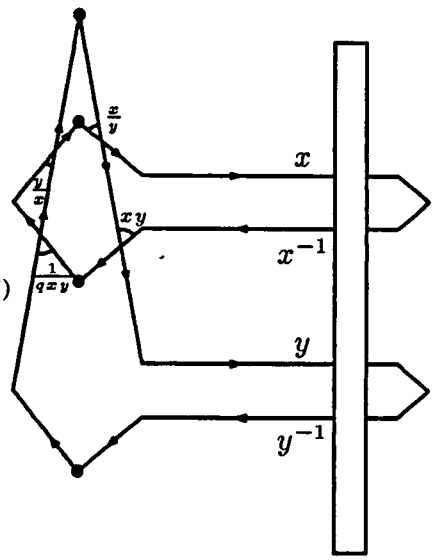
$$\tau(x)\tau(y) =$$



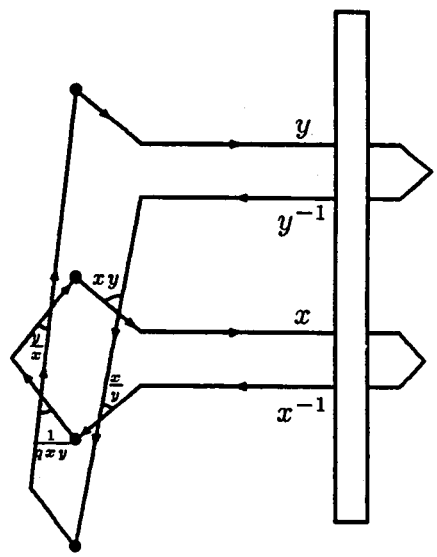
$\overset{(A.6)}{=} \frac{1}{c_1(xy)}$



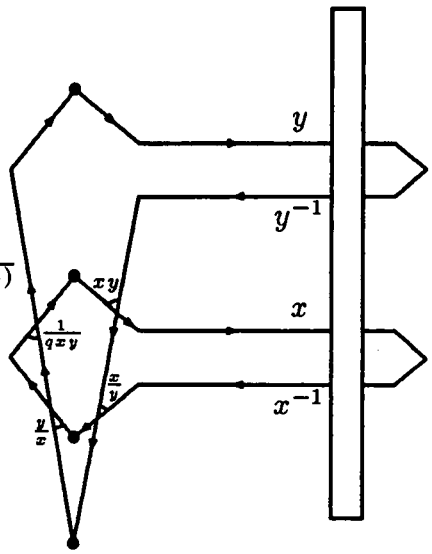
$\overset{(A.7)}{=}$



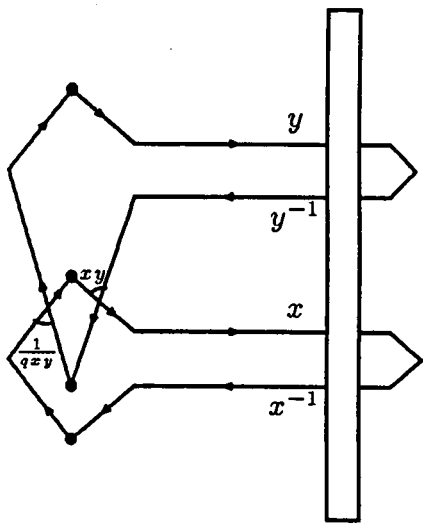
$\overset{(2.16)}{=} \frac{1}{c_1(xy)c_2(\frac{x}{y})}$



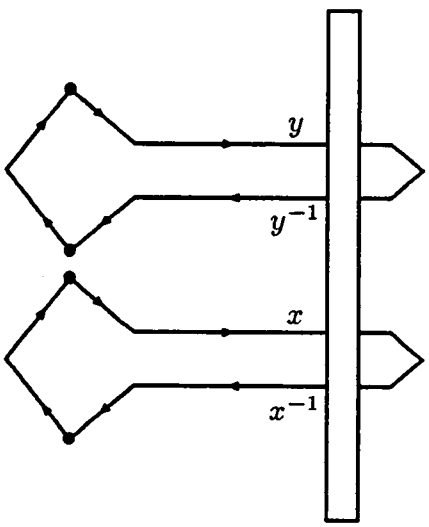
$\overset{(A.9)}{=} \frac{1}{c_1(xy)c_2(\frac{x}{y})}$



$\overset{(A.8)}{=} \frac{c_2(\frac{x}{y})}{c_1(xy)c_2(\frac{x}{y})}$



$=$



$= \tau(y)\tau(x)$

APPENDIX B : BAE for periodic boundary conditions

In this appendix we present the Bethe ansatz equations for a graded 15-vertex model with anisotropy using periodic boundary conditions. The procedure is the same as employed in section 3 of chapter 2, but using the trigonometric form of the S-matrix (eqs.(2.2)). In fact, we can obtain the BAE directly by substituting eqs.(2.2) in eqs.(3.34), (3.35) of chapter 2, resulting in

$$\left(\frac{\sinh\frac{\gamma}{2}(v_j-i)}{\sinh\frac{\gamma}{2}(v_j+i)}\right)^L \prod_{k\neq j}^{N} \frac{\sinh\frac{\gamma}{2}(v_j-v_k+2i)}{\sinh\frac{\gamma}{2}(v_j-v_k-2i)} \times \prod_{\alpha=1}^{M} \frac{\sinh\frac{\gamma}{2}(v_j-\nu_\alpha-i)}{\sinh\frac{\gamma}{2}(v_j-\nu_\alpha+i)} = 1\ , \quad j=1,\ldots,N,$$

$$\prod_{j=1}^{N} \frac{\sinh\frac{\gamma}{2}(\nu_\alpha-v_j-i)}{\sinh\frac{\gamma}{2}(\nu_\alpha-v_j+i)} = 1, \quad \alpha=1,\ldots,M$$

APPENDIX C : Superalgebra $spl(2,1)$

The classical simple graded Lie algebra $spl(2,1)$ is defined as the associative algebra over $\mathbf{C}$ generated by h_i , f_i , $i = 1,2$ (in the "graded" Cartan-Chevalley basis), where h_i , e_1 , f_1 (e_2 , f_2) are even (odd) generators, which satisfy the (anti) commutation relations

$$[h_1, h_2] = 0, \tag{C.1}$$

$$[h_i, f_j] = a_{ij} f_j, \quad [h_i, e_j] = -a_{ij} e_j,$$

$$[f_1, e_1] = h_1, \quad [f_1, e_2] = 0,$$

$$[f_2, e_2]_+ = h_2, \quad [f_2, e_1] = 0,$$

$$[f_2, f_2]_+ = [e_2, e_2]_+ = 0,$$

together with the cubic Serre relations

$$(f_1)^2 f_2 - 2 f_1 f_2 f_1 + f_2 (f_1)^2 = 0 \tag{C.2}$$

$$(e_1)^2 e_2 - 2 e_1 e_2 e_1 + e_2 (e_1)^2 = 0$$

$$f_1 (f_2)^2 - (f_2)^2 f_1 = 0$$

$$e_1 (e_2)^2 - (e_2)^2 e_1 = 0$$

In the relations above a_{ij} is an element of the "graded" Cartan matrix A given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{C.3}$$

The fundamental representation of the generators is given below

$$h_1 \equiv w_1 - w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 \equiv w_2 + w_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{C.4}$$

$$f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The $spl(2,1)$ graded Lie Algebra can also be defined in the Cartan-Weyl basis [43], as in chapter 2.

APPENDIX D : Proof of some q-commutation rules

In this appendix we compute the commutation rule between F_1 and E_1 and the first cubic q-Serre relations in (4.13). We begin by choosing $\alpha = 1$, $\beta = 2$, $\alpha'' = 2$, $\beta'' = 1$ in the Yang-Baxter equation (4.9). We find

$$S_{+12}^{21} T_{+1}^{\ 1} T_{-2}^{\ 2} + S_{+21}^{21} T_{+1}^{\ 2} T_{-2}^{\ 1} = T_{-2}^{\ 1} T_{+1}^{\ 2} S_{+12}^{12} + T_{-1}^{\ 1} T_{+2}^{\ 2} S_{+12}^{21} \qquad (D.1)$$

Using the definition of S_+ (4.2), T_+ (4.4), T_- (4.5) we get

$$\begin{aligned} E_1 q^{\frac{1}{2}W_3} q^{-\frac{1}{2}W_3} F_1 - q^{-\frac{1}{2}W_3} F_1 E_1 q^{\frac{1}{2}W_3} = \\ = \frac{1}{(q-q^{-1})} \left(q^{-W_1+W_2} - q^{W_1-W_2}\right) \end{aligned} \qquad (D.2)$$

To prove that q^{W_3} commutes with $F_1(E_1)$ we repeat the same procedure using $\alpha = 1$, $\beta = 3$, $\alpha'' = 2$, $\beta'' = 3$ ($\alpha = 2$, $\beta = 3$, $\alpha'' = 1$, $\beta'' = 3$) in eq.(4.9) (eq.(4.10)). Then we find

$$[E_1, F_1] = \frac{q^{-H_1} - q^{H_1}}{q - q^{-1}} \qquad (D.3)$$

as already pointed out in section 4.

To demonstrate the first cubic q-Serre relation in (4.13), we begin by noting that this expression can be rewritten as

$$[F_1, [F_2, F_1]_q]_q = 0 \qquad (D.4)$$

where $[A, B]_q \equiv AB - qBA$. The internal q-commutation rule between F_2 and F_1 can be computed by adopting $\alpha = 2$, $\beta = 1$, $\alpha'' = 3$, $\beta'' = 2$ in eq.(4.8). Proceeding along the same lines as in the calculation of $[E_1, F_1]$ we get

$$[F_2, F_1]_q = -\frac{q^{-\frac{1}{2}}}{q - q^{-1}} q^{\frac{1}{2}W_2 + W_3} \tilde{\sigma} T_{+1}^{\ 3} \equiv M_1^3 \qquad (D.5)$$

Finally to show that $[F_1, M_1^3]_q = 0$, we shall use $\alpha = 1$, $\beta = 1$, $\alpha'' = 2$, $\beta'' = 3$ in (4.8). This completes the proof. Similarly, we can demonstrate the other relations.

APPENDIX E : Proof of the $spl_q(2,1)$ invariance of the model

In this appendix we derive the commutation relation between $\mathcal{A}(x)$ and F_1. By choosing $\alpha = 1$, $\beta = 2$, $\gamma = 2$, $\delta = 2$ in the Yang-Baxter equation for the "doubled" monodromy matrix $\mathcal{U}$ (3.2) we obtain

$$\begin{aligned}\mathcal{A}(x)\mathcal{C}_2(x') = &\frac{a(x/x')a(xx')}{b(x/x')b(xx')}\mathcal{C}_2(x')\mathcal{A}(x) - \frac{a(xx')c_-(x/x')}{b(xx')\ b(x/x')}\mathcal{C}_2(x)\mathcal{A}(x') \\ &- \frac{c_-(xx')c_-(x/x')}{b(xx')b(x/x')}\left(\mathcal{D}_2^2(x)\mathcal{C}_2(x') + \mathcal{D}_3^2(x)\mathcal{C}_3(x')\right) \\ &+ \frac{a(x/x')c_-(xx')}{b(x/x')b(xx')}\left(\mathcal{D}_2^2(x')\mathcal{C}_2(x) + \mathcal{D}_3^2(x')\mathcal{C}_3(x)\right)\end{aligned} \tag{E.1}$$

In the limit $x' \to \infty$, according to (2.2), the above expression results in

$$\mathcal{A}(x)\mathcal{C}_2(x' \to \infty) = \mathcal{C}_2(x' \to \infty)\mathcal{A}(x) + q(q - q^{-1})\mathcal{C}_2(x)\mathcal{A}(x' \to \infty) \tag{E.2}$$

Then by using (4.15) we have

$$\mathcal{A}(x)q^{-\frac{1}{2}W_3}F_1 q^{W_1} = q^{-\frac{1}{2}W_3}F_1 q^{W_1}\mathcal{A}(x) + q^{\frac{3}{2}-\frac{L}{2}}\mathcal{C}_2(x)q^{2W_1} \tag{E.3}$$

Multiplying from the left to the right by $q^{\frac{1}{2}W_3}$ and from the right to the left by q^{-W_1} we get

$$\underbrace{q^{\frac{1}{2}W_3}\mathcal{A}(x)q^{-\frac{1}{2}W_3}}_{\mathcal{A}(x)}F_1 = F_1\underbrace{q^{W_1}\mathcal{A}(x)q^{-W_1}}_{\mathcal{A}(x)} + q^{\frac{3}{2}-\frac{L}{2}}q^{\frac{1}{2}W_3}\mathcal{C}_2(x)q^{W_1} \tag{E.4}$$

To prove that $\mathcal{A}(x)$ commutes with q^{W_3} (q^{W_1}) we repeat the same strategy using $\alpha = 1$, $\beta = 3$, $\gamma = 1$, $\delta = 3$ ($\alpha = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$) in (3.2) for the limit $x' \to 0(\infty)$. Therefore, we find

$$[\mathcal{A}(x), F_1] = q^{\frac{3}{2}-\frac{L}{2}}q^{\frac{1}{2}W_3}\mathcal{C}_2(x)q^{W_1} \tag{E.5}$$

as already pointed out in sec. 4. All other relations (4.16)-(4.19) are computed analogously.

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KURZFASSUNG DER ERGEBNISSE

In der vorliegenden Arbeit wurde das eindimensionale supersymmetrische t-J Modell untersucht. Dieses Modell beschreibt ein System von stark wechselwirkenden Elektronen auf einem eindimensionalen Gitter. Durch die Anwendung der algebraischen "nested Bethe ansatz"-Methode habe ich die Eigenvektoren und die Eigenwerte konstruiert. Außerdem habe ich eine neue Form der Bethe-Ansatz-Gleichungen erhalten.

Weiter wurden die algebraischen Eigenschaften des Modells ausführlich diskutiert. Es wurde gezeigt, daß die Bethe-Ansatz-Zustände höchste-Gewichts-Zustände der $spl(2,1)$ Superalgebra sind. Dann habe ich durch die Anwendung der "shift" Operatoren über die Bethe Zustände einen vollständigen Satz von Bethe-Vektoren erhalten.

Es wurde auch die Grundzustandsenergie und das Energiespektrum der elementaren Anregungen, wie "spinons" und "holons" bestimmt. Die Dispersionsrelationen, die Fermi-Kanten, die magnetische Suszeptibilität als Funktion des magnetischen Feldes, sowie die Elektrondichte als Funktion des chemischen Potentials wurden ebenfalls berechnet.

Es wurde weiterhin ein verallgemeinertes anisotropes t-J Modell eingeführt. Durch die Anwendung von Markov-Spuren wurde eine Methode entwickelt, um spezielle Randbedingungen zu behandeln. Es wurde gezeigt, daß dieses q-deformierte t-J Modell mit diesen Randbedingungen $spl_q(2,1)$-invariant ist.

Das im Vergleich zum Fall von periodischen Randbedingungen wesentlich kompliziertere Problem, die Bethe-Ansatz-Gleichungen herzuleiten wurde durch eine Erweiterung der Methode des algebraischen "nested Bethe ansatz"gelöst. Mit Hilfe der Yang-Baxter-Algebra wurde eine explizite Realisierung der Quantengruppe $spl_q(2,1)$ konstruiert.

DANKSAGUNG

Herrn Dr. Michael Karowski möchte ich ganz herzlich für die interessante Themenstellung, die hervorragende Betreuung, sowie zahlreiche Diskussionen und Anregungen danken.

Mein Dank gilt auch Herrn Prof. Dr. Robert Schrader für die Gelegenheit, in seiner Arbeitsgruppe mitarbeiten zu können.

Weiterhin danke ich allen Mitgliedern der Arbeitsgruppen Schrader, Kleinert, Peschel und Schotte für ein angenehmes Arbeitsklima.

Bei dem Deutschen Akademischen Austauschdienst bedanke ich mich für die finanzielle Unterstützung.

Mein herzlicher Dank gilt meinem Mann Pedro. Er hat mich auf jede Art unterstützt und so entscheidend zum Gelingen dieser Arbeit beigetragen.

LEBENSLAUF

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