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SAMPLED-DATA CONTROL OF LINEAR SYSTEMS SUBJECT TO INPUT SATURATION: A HYBRID SYSTEM APPROACH

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DEDICATÓRIA

Dedico este trabalho aos meus pais.

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ABSTRACT

In this work, a new method for the stability analysis and synthesis of sampled-data control systems subject to variable sampling intervals and input saturation is proposed. From a hybrid systems representation, stability conditions based on quadratic clock-dependent Lyapunov functions and the generalized sector condition to handle saturation are developed. These conditions are cast in semidefinite and sum-of-squares optimization problems to provide maximized estimates of the region of attraction, to estimate the maximum intersampling interval for which a region of stability is ensured, or to produce a stabilizing controller that results in a large implicit region of attraction, through the maximization of an estimate of it.

Keywords: Sampled-data systems, stability and stabilization, actuator saturation, hybrid systems, semidefinite programming, sum of squares.

RESUMO

Neste trabalho é proposto um novo método para a análise da estabilidade de sistemas de controle amostrados aperiodicamente e com saturação na entrada, e também para a síntese de controladores estabilizantes. A partir de uma representação por sistemas híbridos, condições de estabilidade baseadas em funções quadráticas de Lyapunov dependentes do *clock* e na condição de setor generalizada para o tratamento de saturação são desenvolvidas para o sistema amostrado em questão. Essas condições são incorporadas como restrições em problemas de otimização. Os problemas de otimização são baseados em programação semidefinida e em programação *sum-of-squares*, e têm o objetivo de obter estimativas maximizadas da região de atração do sistema, estimativas do intervalo de amostragem máximo para o qual uma dada região de estados iniciais seja uma região de estabilidade, ou para produzir controladores (dados por ganhos estáticos estabilizantes) que resultem em uma região de atração implicitamente grande, através da maximização da estimativa dessa região de atração.

Palavras-chave: Sistemas amostrados, estabilidade e estabilização, saturação no atuador, sistemas híbridos, programação semidefinida, sum of squares.

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LIST OF ABBREVIATIONS

LMI	Linear matrix inequality
LTI	Linear time-invariant (system)
MIMO	Multiple-input multiple-output
SDP	Semidefinite programming
SDS	Sampled-data system
SISO	Single-input single-output
SOS	Sum of squares
SOSP	Sum-of-squares programming
UMP	Univariate matrix polynomial

LIST OF SYMBOLS

$\sum_{i=a}^{b} x_i$	Sum of $x_a, x_{a+1}, \cdots, x_{b-1}, x_b$
$\sum_{a \le i \le b} x_i$	Sum of $x_a, x_{a+1}, \cdots, x_{b-1}, x_b$
$\sum_{a \le i_1 + \cdots + i_z \le b} x_{\{i_1, \cdots, i_z\}}$	Sum of every possible $x_{\{i_1, \cdots, i_z\}}$ with $a \le i_1 + \cdots + i_z \le b$
$A \Rightarrow B$	A is sufficient for B / B is necessary for A;
$A \Leftarrow B$	A is necessary for B / B is sufficient for A;
$A \Leftrightarrow B$	A is equivalent to B;
$c \in C$	c is an element of the set C;
$C \subset D$	C is a subset of D;
∂D	The boundary of the set D i.e. the whole D minus its interior;
$\forall$	For all;
$\mathbb{N}$	The set of the natural numbers;
$\mathbb{N}^+$	The set of the positive natural numbers;
$\mathbb{R}$	The set of the real numbers;
$\mathbb{R}^+$	The set of the positive real numbers;
$\mathbb{R}^n$	n-dimensional real space;
$v \in \mathbb{R}^n$	v is a vector of dimension n;
$v_{(i)}$	The *i*th element of the vector v;
$\mathbb{R}^{n \times n}$	nxn-dimensional real space;
$M \in \mathbb{R}^{n \times n}$	M is a matrix of dimension $n \times n$;
$M_{(i)}$	The *i*th row of the matrix M;
$M_{(i)}$	The element in the *i*th row and *j*th column of the matrix M;
M'	The transpose of M;
$\mathbb{S}^n$	The space of all symmetric matrices of dimension $n \times n$;
$S \in \mathbb{S}^n$	S is a symmetric matrix of dimension $n \times n$;
$S > 0$	S is positive definite;

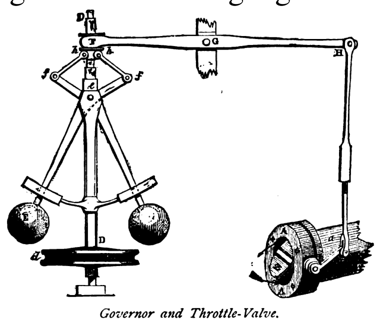
$S \geq 0$	S is positive semidefinite;
diag$(s_1, \cdots, s_m)$	The diagonal matrix of blocks $s_1, \cdots, s_m$
I	An identity matrix of adequate dimension;
$\dot{f}(x)$	The derivative of $f(x)$;
$\nabla f(x)$	The gradient of a $f(x)$. If the function is $f\ :\mathbb{R}^n \to \mathbb{R}$, it is given by the identity $\nabla f(x) = [\frac{df}{dx_{(1)}}(x), \cdots, \frac{df}{dx_{(n)}}(x)]$;
$x^+(t)$	x in the instant after t, i.e. $x^+(t) = \lim_{\tilde{t}\to t} x(\tilde{t})$, with $\tilde{t} > t$;
$a \to b$	the value of $a \in \mathbb{R}$ converges to the value of $b \in \mathbb{R}$, i.e. for an arbitrarily small $\epsilon > 0$, it is verified that $\lvert a - b \rvert < \epsilon$.

CONTENTS

1 INTRODUCTION

In present times, there are many instances of dynamical systems that are recognized as control systems. Countless examples from the industries of automobiles and spacecraft, robot manufacturers, and from the process industry could be given here. Since the beginning of control theory, which could arguably be considered when James Clerk Maxwell started to develop a mathematical model for the centrifugal governor of Figure 1 in the 19th century, all of the listed examples have shaped the ramifications of control theory applied in industry, attracting its focuses of research and thus dividing it in several branches. And although the groundwork needed for the advances of each branch of the applied theory is reduced by the overlapping aspects of the real applications, there always arise other opportune aspects to expand the boundaries of the state of art.

Figure 1 – Centrifugal governor.



Source: MUIR, 1875

Such is the case of the *networked control systems*. When the elements of a control system are separated by a network, there arise some particularities that are not foreseen by the classical theory, such as the loss of periodicity caused by congestion and eventual packet drop-outs (ANTSAKILS; BAILLIEUL, 2007). By "periodicity" is expressed the uniform rate at which the signals from the plant in this system would ideally be sampled, and read by the controller. The fact that some network protocols offer no guarantee of a uniform sampling rate is one of the issues of this type of system, and alone is the motivation for the research on the stability of sampled-data systems with aperiodic sampling (HETEL *et al.*, 2016).

A sampled-data system (SDS), in this context, is viewed as a simplified representation of the networked control topology. It represents one continouous-time plant and one digital controller interconnected as if they were separated by a network. Hence, the stability of sampled-data systems has been receiving a lot of attention since it has first been associated to the networked control, and numerous authors have shown concern over the issue in the last years (FRIDMAN, 2010, SEURET, 2012, FIACCHINI; GOMES DA SILVA Jr., 2018, BRIAT, 2013, HETEL *et al.*, 2016). While the aperiodic sampling is common to all of their approaches, their differences include (and are not limited to) the mathematical representation of the sampled-data system and the theoretical tools for the stability analysis.

The stability analysis of a SDS can be performed in different ways, depending on the problem complexity. The stability of a linear and periodic SDS, for a simple instance, can be indirectly evaluated in a discrete model obtained by an exact discretization of the SDS continuous dynamics i.e. evaluated from a discrete model whose state values coincide to the ones of the SDS at the periodic sampling instants. And the evaluation of stability is global in this case, i.e. it is valid for all the domain of the plant state. When, adversely, the SDS sampling is not periodic, the system can no longer be discretized[1] in an exact model and the indirect evaluation is no longer valid; instead, it becomes necessary to consider explicitly that the plant state evolves continuously between samplings, and that the control signal, contrarily, remains constant between samplings, only being updated at the sampling instants. Still, the stability of a SDS remains globally defined in this aperiodic case.

More adversely, if the SDS is affected by input saturation, then, making things worse, the stability of the equilibrium point is not necessarily global. A saturating actuator degrades the performance of a control system as much as the trajectories of its state evade what is called *the linear region* i.e. the region where the actuator is not saturated. When it is stated that the stability of a SDS is not global, it may as well be said that the stability of such system is verified only locally in the *region of attraction* (that is associated to the origin of the SDS), beyond which the degradation accrued from the saturation culminates in the instability of the trajectories. In the analysis of stability, any subset providing an estimate of this region is named a *region of stability*. The estimates are more easily obtained when confined in the linear region, but can be extended past its boundaries with the use of, for instance, polytopic differential inclusions, piecewise affine representations or *the generalized sector condition* (see GOMES DA SILVA Jr.; TARBOURIECH, 2005). In particular, the generalized sector condition technique has been used in the context of aperiodic sampling combined with saturating actuators in the works of SEURET; GOMES

[1]It is actually possible to obtain a discrete-time uncertain system, as was done in FIACCHINI; GOMES DA SILVA Jr., 2018, but the affirmation interjected by this side note remains valid, nonetheless.

DA SILVA Jr., 2012 and GOMES DA SILVA Jr. *et al.*, 2016.

In this study, sufficient conditions for the stability analysis and synthesis of a SDS subject to aperiodic sampling and input saturation are proposed. A first consideration is that the hybrid nature of the continuous-time plant and discrete-time controller that compound the SDS evokes the dynamics of a hybrid system (GOEBEL; SANFELICE; TEEL, 2012). The continuous and discrete dynamics of this system interplay according to an auxiliary variable, named the clock, that marks the incidence of each dynamic in the always advancing time. Stability conditions based on a quadratic clock-dependent Lyapunov function (BRIAT, 2015) and on the generalized sector condition are developed for the hybrid system representation. The conditions are cast in semidefinite and sum-of-squares optimization problems to: (P1) provide the largest estimates of the region of attraction; (P2) estimate the maximum intersampling interval for which a region of stability is ensured; (P3) produce a stabilizing controller that results in a large implicit region of attraction, through the maximization of an estimate of it.

The outline of this dissertation is as follows:

In Chapter 2, the stability analysis of sampled-data systems is reviewed in the literature, with a particular emphasis on works dealing with actuator saturation. The approaches of each group of works from the literature are highlighted, with the intent of a characterization of the present study inside its field of research.

In Chapter 3, a hybrid representation of the SDS with intersampling times residing in a closed interval and with input saturation system is formalized. After the representation is presented, the problems P1, P2 and P3 are formally stated, and, aiming for their solution, preliminary conditions based on the clock-dependent Lyapunov function and the generalized sector condition are derived.

In Chapter 4, the problems P1 and P2 are approached with the conditions for stability analysis. A hypothesis is made that the clock dependency of the Lyapunov function is either affine or polynomial, allowing the preliminary stability conditions to be cast in semidefinite and sum-of-squares optimization problems. As examples, some numerical simulations are executed after the optimization problems are solved by appropriate software.

In Chapter 5, the design problem P3 is tackled by the proposition of conditions to compute stabilizing controllers. The preliminary conditions of Chapter 3 are modified through the application of Finsler's lemma with an appropriate structure of multipliers to cast the solution of problem P3 as semidefinite and sum-of-squares programmings. More numerical simulations are provided as examples, to illustrate the methodology.

Additionally, the Annex provides some basic theoretical tools. Along the reading, it will be referenced when its content might be helpful.

2 LITERATURE REVIEW: STABILITY ANALYSIS OF SAMPLED-DATA SYSTEMS WITH INPUT SATURATION

2.1 Introduction

In the present chapter, we cover some studies that have explored the topics within the scope of this dissertation. This chapter is outlined by the major themes of the referenced studies: in Section 2.2, it is attempted to group the references according to what model they used to represent a sampled-data system, and what was the approach for the analysis of the model; next, Section 2.3 considers the saturation effect, and overviews the main concepts used to handle models subject to this nonlinearity; lastly, Section 2.4 exposes the references that have faced the issue of saturation in a sampled-data system, and presents a summary with the overlapping aspects of their approaches.

2.2 Sampled-data systems

If one tries to divide all dynamical systems in those evolving continuously in time and those of discrete nature, his or her attempt may be frustrated by one of the numerous exceptions for this classification. It could be argued that most industrial processes benefiting from the application of control techniques have continuous dynamics; for instance, classical mechanical systems and analog electronic circuits evolving in time according to principles of physics, such as Newton's and Kirchoff's laws, are viewed naturally as continuous-time dynamical systems; it could, likewise, be affirmed that there are systems that can accurately be modelled in discrete-time, such as digital controllers.

There exist, however, dynamical systems that qualitatively manifest the dynamics of both continuous and discrete-time models, and, in the words of GOEBEL; SANFELICE; TEEL, 2012, escape such a clear-cut classification. As similarly reasoned by the authors, examples are provided by stock markets, where the assets prices slowly change but happen to spike sometimes, and by mechanisms whose moving parts are prone to impact. This type of systems can only be accurately represented by a hybrid class of models. Actually, a very large number of systems belong to this class, where abrupt changes in the system state are expected - in fact, any interface of a continuous-time and a discrete-time system could be added to our list of examples. Anyhow, however large the variety of these systems, in the scope of this study we are only interested in the representation of *sampled-data systems*. In this study, the sampled-data systems (SDS) are represented by *hybrid systems* (GOEBEL; SANFELICE; TEEL, 2012), which is exposed among other models for the SDS in the next section.

2.2.1 Models for the sampled-data system

2.2.1.1 Hybrid systems

A hybrid system is formally described by the following model:

$$\begin{cases} \dot{x}(t) = f(x(t)), & \forall x(t) \in C \\ x^{+}(t) = g(x(t)), & \forall x(t) \in D \end{cases} \tag{1}$$

where C and D are respectively called the *flow* and *jump* sets, f and g are respectively called the flow and jump functions, and $x : \mathbb{R}^{+} \to \mathbb{R}^{n}$ is the state variable. The unusual notation x^{+} indicates that $x^{+}(t) = g(x(t))$ is a difference equation, whose solution is a discontinuous function of time. The reviewed literature presents different descriptions of this difference equation: the references adopting models of impulsive differential equations, often called *impulsive systems*, set the discontinuity of the function on the limits of the time interval between the impulse instants, for instance. To be more precise, an impulsive system is expressed below in terms of a set $\Theta = \{t \in \mathbb{R}^{+} : t = t_i,\ i \in \mathbb{N}^{+}\}$ containing the time sequence of impulsive updates in the state variable:

$$\begin{cases} \dot{x}(t) = f(x(t)), & \forall t \notin \Theta; \\ x(t) = g(x(t^{-})), & \forall t \in \Theta; \quad t \neq 0 \\ x(0) = x_0 & t = 0 \end{cases} \tag{2}$$

where $x(t^{-}) = \lim_{\tilde{t} \to t,\ \tilde{t} < t} x(\tilde{t})$. The updates of the state variable in this model can correspond to the samplings in a SDS, which will be detailed soon. This representation was used in some of the references (FIACCHINI; GOMES DA SILVA Jr., 2018 and BRIAT, 2013 and NAGHSHTABRIZI; HESPANHA; TEEL, 2008), with specific functions f and g. In NAGHSHTABRIZI; HESPANHA; TEEL, 2008, for instance, the impulsive system is specified to a LTI system with an extended state variable $x : \mathbb{R}^{+} \to \mathbb{R}^{n = n_p + m}$ given by $x(t) = \left[x_p'(t)\ u'(t)\right]'$:

$$\begin{cases} \dot{x}(t) = A_f x(t) & \forall t \notin \Theta; \\ x(t) = \bar{A}_j x(t^{-}) & \forall t \in \Theta; \quad t \neq 0 \\ x(0) = x_0 & t = 0 \end{cases} \tag{3}$$

where $x_p : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_p}$ is the plant state variable, $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is the control signal, and the matrices A_f and $\bar{A}_j \in \mathbb{R}^{n \times n}$ are given by

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_j = \begin{bmatrix} I & 0 \\ BK & 0 \end{bmatrix}$$

In parallel to hybrid sytems, there is the approach to represent a SDS by *time-delay differential equations*.

2.2.1.2 Time-delay systems

In this model of a SDS, the aperiodic sampling of the SDS is translated into a time-varying delay that keeps the input signal constant between two consecutive samples, allowing the sampled-data system to be represented by an infinite-dimensional model that is expressed by

$$\begin{cases} \dot{x}_p(t) = Ax_p(t) + Bu(t), & \forall t \notin \Theta \\ u(t) = u(t - \tau(t)) = Kx_p(t - \tau(t)), & \forall t \in \Theta; \quad t \neq 0; \\ \tau(t) = t - t_i, & \forall t \in [t_i, t_{i+1}) ; \quad i \in \mathbb{N}^+ \\ x_p(0) = x_{p0}; \quad u(0) = Kx_{p0} & t = 0 \end{cases} \tag{4}$$

where $x_p : \mathbb{R}^+ \rightarrow \mathbb{R}^{n_p}$ is the plant state variable, $u : \mathbb{R}^+ \rightarrow \mathbb{R}^m$ is the control signal, $\tau : \mathbb{R}^+ \rightarrow [0, \overline{\tau}]$ is the delay, and $A \in \mathbb{R}^{n_p \times n_p}$, $B \in \mathbb{R}^{n_p \times m}$, $K \in \mathbb{R}^{m \times n_p}$ are matrices with respect to the plant, the input, and the controller gain, respectively. Notice that $\dot{\tau}(t) = 1$, and that $\tau(t)$ is reset to zero at each sampling instant t_i.

The difference between the models of a SDS (time-delay and hybrid systems) presented until now is that in hybrid systems, there is a function (the jump function) mapping the update in the control signal (and in the plant state as well, that remains constant) at the sampling times, whereas in the time-delay representation, there is no function mapping the control signal as if it were a variable; instead, the control signal itself is a function that depends on the plant and controller state delayed in time. One implication of this difference is that the hybrid system is finite-dimensional, and the time-delay system is infinite-dimensional.

2.2.2 Stability analysis of the sampled-data system

The stability of each model of the SDS can be qualitatively evaluated by two analytical tools that are derived from the Lyapunov notion of stability: the *Lyapunov-Krasovskii theorem* (KOLMANOVSKII; MYSHKIS, 1992) theorem and the *clock-dependent Lyapunov functions* theorem (GOEBEL; SANFELICE; TEEL, 2009). Although different, these theorems share the same idea in reducing the analysis conservatism by generalizing the classical Lyapunov function candidate. The Lyapunov-Krasovskii theorem is used to evaluate the stability of time-delay systems and underpins the *time-delay approach*; on the other hand, the *hybrid system approach* is based on the theorem of clock-dependent Lyapunov functions to evaluate hybrid systems. The next sections review a part of the literature according to the approach used.

2.2.2.1 Time-delay approach

In LI; DE SOUZA, 1997, stability conditions were developed for the problem of a plant with uncertain parameters and time-varying-delay, and were based on the Lyapunov-Razumikhin theorem for an unrestricted delay. In the study of FRIDMAN; SEURET; PIERRE RICHARD, 2004, the delay is restricted to evolve at the same rate of time ($\dot{\tau}(t) = 1$, which is equivalent to say that $\tau(t)$ is given by the corresponding equation in (4)). The restriction was proposed to reduce the conservatism of the results based on the Lyapunov-Razumikhin theorem *without* a loss of generality, since the restriction on $\dot{\tau}$ is *intrinsic* to the SDS. The stability conditions in this case are based in the Lyapunov-Krasovskii theorem, and the existence of a Lyapunov-Krasovskii Functional (LKF) satisfying the conditions is the argument that certifies the stability of system (4).

Later, SEURET, 2012 modifies the stability conditions by exchanging the positive definiteness condition on the LKF with a so called looping-condition, and consequently establishes the concept of the *looped-functionals*. The looped-functionals are used to evaluate the stability of system (4) with the stability conditions (among which is the looping-condition) that are presented in Theorem 1:

Theorem 1. *(adapted from SEURET, 2012) Let $c_1 < c_2$ be two positive scalars, $\mathcal{D}_{xp} \subset \mathbb{R}^{n_p}$ be the domain of the differentiable function $V : \mathcal{D}_{xp} \to \mathbb{R}^+$, and $\mathcal{E}_{Vp} \subset \mathcal{D}_{xp}$ be its subset containing the equilibrium point $x_p = 0$. Let $\mathbb{K}$ be a set of differentiable functions, $\mathcal{V}_0 : \mathbb{R}^+ \times \mathbb{K} \to \mathbb{R}$ be a differentiable functional, and $z \in \mathbb{K}$ be some function. If the following conditions are verified*

$$c_1||x_p(t)||^2 \leq V(x_p(t)) \leq c_2||x_p(t)||^2 \quad \forall x_p(t) \in \mathcal{D}_{xp}$$

$$\mathcal{V}_0(t_{i+1}, z(\cdot)) = \mathcal{V}_0(t_i, z(\cdot)) \quad \forall t_i, t_{i+1} \in \Theta, \quad i \in \mathbb{N}^+ \tag{5}$$

$$\frac{d}{dt}\mathcal{W}_0(x_p(t), t, z(\cdot)) = \frac{d}{dt}\left(V(x_p(t)) + \mathcal{V}_0(t, z(\cdot))\right) < 0 \quad \forall x_p(t) \in \mathcal{D}_{xp} \tag{6}$$

then $\mathcal{E}_{Vp}$ is a region of asymptotic stability of system (4)*, i.e, given that $x_p(0) \in \mathcal{E}_V$ and that* (4) *has solution, it follows that $x_p(t) \to 0$ as $t \to \infty$*

In his study, SEURET particularly uses a quadratic function V along with a functional candidate $\mathcal{V}_0$ that are given by (7), where P, S_1, S_2, R, and X are matrices of appropriate dimensions.

$$\begin{cases} V(x_p(t)) = x_p'(t)Px_p(t) \\ \mathcal{V}_0(t, x_p(\cdot)) = (t_{i+1} - t)(V_1(t, x_p(\cdot)) + V_2(t, x_p(\cdot)) + V_3(t, x_p(\cdot))) \qquad \forall t \in [t_i,\ t_{i+1}] \\ V_1(t, x_p(\cdot)) = x'(t)(S_1 x_p(t) + 2S_2 x_p(t_i)) \\ V_2(t, x_p(\cdot)) = \int_{t_i}^{t} \dot{x}_p(\theta) R \dot{x}_p(\theta) d\theta \\ V_3(t, x_p(\cdot)) = (t - t_i) x_p'(t_i) X x_p(t_i) \end{cases} \tag{7}$$

2.2.2.2 Hybrid system approach

Here is made explicit one consideration that all studies referenced in this section and the last have taken into account: it is considered that the time interval between samplings t_i and t_{i+1} is restricted to an interval bounded by $\underline{\tau}$ and $\overline{\tau}$, to represent the minimum and maximum intersampling times observed in the SDS, respectively. In more precise words, it is considered that $t_{i+1} - t_i \in [\underline{\tau},\ \overline{\tau}]$, $\forall i \in \mathbb{N}^+$, in an assumption that, at least, the extremes of the admissible intervals of sampling are known.

With this information and returning some years in the timeline to the already cited study of (NAGHSHTABRIZI; HESPANHA; TEEL, 2008), the impulsive system (3) is analysed as well with a particular functional $\mathcal{V}_0$ that results from a sum of parts quadratic on x_p, as follows:

$$\begin{cases} \mathcal{V}_0(t, x_p(\cdot), x(t)) = V_1(x_p(t)) + V_2(x(t), t) + V_3(t, x_p(\cdot)) \qquad \forall t \in [t_i,\ t_{i+1}] \\ V_1(t, x_p(\cdot)) = x_p'(t)Px_p(t) \\ V_2(t, x(t)) = x'(t) \left(\int_{t_i - t}^{t_i} (\theta + \overline{\tau})(A_f e^{A_f \theta})' \tilde{R} (A_f e^{A_f \theta}) d\theta \right) x(t) \\ V_3(t, x_p(\cdot)) = (\overline{\tau} - (t - t_i))'(x_p(t) - x_p(t_i))' X (x_p(t) - x_p(t_i)) \\ \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \end{cases} \tag{8}$$

Unlike the one shown in the time-delay approach, however, this functional is not based on the Lyapunov-Krasovskii theorem or, for that matter, on Theorem 1. Rather, the already mentioned theorem of clock-dependent Lyapunov functions, from GOEBEL; SANFELICE; TEEL, 2009, underpins the current approach. This theorem, differently from Theorem 1, enunciates functions as $V : \mathcal{D}_x \times [0,\ \overline{\tau}] \to \mathbb{R}^+$, where $\mathcal{D}_x \subset \mathbb{R}^n$, and for this reason the functional (8) must be modified. It is considered as modification that $\mathcal{V}_0$ is not a function of $x_p(\cdot)$, and that $x_p(t_i)$ is a "global variable", or in other words a parameter not included in the list of arguments of the function. This first consideration demotes $\mathcal{V}_0$ from the title of functional. A second consideration is that $\mathcal{V}_0$ is not a function of t, but of $\tau(t)$. After these considerations, the new and yet equivalent function candidate is given by:

$$\begin{cases} V(x(t),\tau(t)) = V_1(x(t)) + V_2(x(t),\tau(t)) + V_3(x(t),\tau(t)) & \forall t \in [t_i,\ t_{i+1}] \\ V_1(x(t)) = x_p'(t)Px_p(t) \\ V_2(x(t),\tau(t)) = x'(t)\left(\int_{-\tau(t)}^{0}(\theta+\overline{\tau})(A_f e^{A_f\theta})'\tilde{R}(A_f e^{A_f\theta})d\theta\right)x(t) \\ V_3(x(t),\tau(t)) = (\overline{\tau}-\tau(t))'(x_p(t)-x_p(t_i))'X(x_p(t)-x_p(t_i)) \\ \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \end{cases}$$

Consider again $c_2 > c_1$, a subset $\mathcal{E}_V \subset \mathcal{D}_x$ and a function $V : \mathcal{D}_x \times [0,\ \overline{\tau}] \to \mathbb{R}^+$ that satisfy the following inequalities:

$$c_1||x(t)||^2 \leq V(x(t),\tau(t)) \leq c_2||x(t)||^2 \qquad \forall x(t) \in \mathcal{D}_x, \quad \forall \tau(t) = [0,\ \overline{\tau}]$$

$$\nabla V(x(t),\tau(t))A_f x(t) < 0 \qquad \forall x(t) \in \mathcal{D}_x, \quad \forall \tau(t) = [0,\ \overline{\tau}) \tag{9}$$

$$V(\bar{A}_j x(t),0) - V(x(t),\tau(t)) < 0 \qquad \forall x(t) \in \mathcal{D}_x, \quad \forall \tau(t) = [\underline{\tau},\ \overline{\tau}] \tag{10}$$

The existence of such function certifies, as in Theorem 1, that $\mathcal{E}_V$ is a region of stability. Taking into account the quadratic function $V(x(t),\tau(t)) = x(t)P(\tau(t))x(t)$, some years earlier BRIAT, 2013 used this particular candidate to propose stability conditions for the aperiodic SDS. This may be considered a starting point for what will be developed in this work: the same candidate is considered, in a hybrid approach (as the title suggests), to propose stability conditions for an aperiodic SDS. The crucial difference of this study to the one of BRIAT is the feature of input saturation, which will be detailed in the next section. As a last remark, the study of BRIAT, 2015, published some years later, is cited here for providing some links between the looped-functional and the hybrid approaches.

2.3 Linear systems with input saturation

The saturation nonlinearity is an ubiquitous feature of real control systems. In practice, a sensor will always have its measurement limited to a range, and an actuator will always have its output amplitude limited by construction or safety reasons. Formally, the scalar saturation function can be described by:

$$\text{sat}(v) = \begin{cases} v_{max}, & \text{if} \quad v > \overline{v} \\ v, & \text{if} \quad \overline{v} > v > \underline{v} \\ v_{min}, & \text{if} \quad \underline{v} > v \end{cases}$$

Input saturation can lead a linear closed-loop system to performance degradation, to the occurrence of limit cycles, multiple equilibrium points and even to instability (TARBOURIECH *et al.*, 2011). There are basically three main approaches to deal with the saturation effects in a control system: the first is to compute the controller considering explicitly the saturating limits of the control signal (TARBOURIECH *et al.*, 2011,HU; LIN, 2001); the second is to utilize *anti-windup* compensators, introducing an extra feedback loop that mitigates the windup in controller during saturation, considering a pre-computed nominal controller (ZACCARIAN; TEEL, 2011); the third is based on *model predictive control*, where the model is used to predict the system output affected by saturation, and a sequence of controller signals is iteratively calculated to optimize a performance criterion taking into account control limitation (CAMACHO; BORDONS ALBA, 2007, MACIEJOWSKI, 2000). The present study fits the first approach.

In TARBOURIECH *et al.*, 2011, numerous perspectives and analysis methods for linear systems subject to input saturation are detailed. Particularly relevant to this study is the one therein that models saturation by the difference of a linear and a deadzone function. The *generalized sector condition*, devised for a less conservative analysis of these systems than the one provided by the classical approach (see GOMES DA SILVA Jr.; TARBOURIECH, 2005 for this comparison), can be applied to derive local (regional) stability conditions.

To see this matter from the viewpoint of the problem that it brings instead of the methodology used to solve it, consider the continuous-time plant described by the following linear time-invariant model:

$$\dot{x}_p(t) = Ax_p(t) + Bu(t)$$

To represent feedback by a static controller, the control signal is described as a function of the system state $u(t) = Kx_p(t)$, where K is the controller static gain. The closed-loop system can be written as

$$\dot{x}_p(t) = (A + BK)x_p(t),$$

and its asymptotic stability is globally characterized by the eigenvalues of $(A + BK)$.

Although the origin of a linear closed-loop system is supposed to be either globally asymptotically stable or not stable in the absence of input constraints, this is not the case when saturation is present. Consider the next system

$$\dot{x}_p(t) = Ax_p(t) + B\text{sat}(Kx_p(t)) \tag{11}$$

In the case of control systems given by the expression above, the stability of the origin is not necessarily global, as the system is nonlinear. Thus, it is important to characterize the *region of attraction* of the origin of (11).

In what follows, the region of attraction to the origin of a system subject to input saturation is defined.

Definition 1. *(TARBOURIECH et al., 2011) The region of attraction R_A of the system* (11) *is defined as the set of all points $x \in \mathbb{R}^n$ of the state space for which $x_p(0)$ leads to* *a trajectory $x_p(t)$ that converges asymptotically to the origin. In other words, if $x_p(0) \in R_A$, then $x_p(t) \to 0$ as $t \to +\infty$.*

The studies that deal with the stability analysis of systems under saturation have the goal of unveiling its region of attraction. A straightforward method to almost exactly determine the region of attraction of any time-invariant system is the one of performing an extensive number of simulations until the geometry of the region of attraction can be satisfyingly delimited with the trajectories that do not diverge from the origin. Unfortunately, the usefulness of this method is lost when there is a high number of states in the system. Indeed, the significant attention that the stability analysis receive from researchers is in part because it is not easy to delimit the exact region of attraction of most nonlinear systems, or even systems that are linear by parts only (see this task undertaken in ROMANCHUK, 1996, where a convex polytope is used to characterize the region of attraction of a piece-wise linear system). Alternatively, there are significant advantages in searching for subsets of the region of attraction that are defined by analytical expressions. These subsets are called *regions of stability*.

In the case of systems with input saturation, the characterization of the region of stability is very often based on the Lyapunov definition of stability. This definition, which relies on the existence of a function $V : \mathcal{D}_{xp} \to \mathbb{R}^+$, is formalized in the following theorem:

Theorem 2. *(KHALIL,1996) Let $x_p : \mathbb{R}^+ \to \mathbb{R}^{n_p}$ be the state vector of the system* (11)*, $\mathcal{D}_{xp} \subset \mathbb{R}^{n_p}$ be a domain containing $x_p = 0$ in its interior, and $\mathcal{E}_{Vp} \subset \mathcal{D}_{xp}$ be its subset also containing $x_p = 0$. Let $V : \mathcal{D}_{xp} \to \mathbb{R}^+$ be a continuously differentiable function, and c_1 and c_2 nonnegative constants such that*

$$c_1||x_p(t)||^2 \leq V(x_p(t)) \leq c_2||x_p(t)||^2 \quad \forall x_p(t) \in \mathcal{D}_{xp}$$

$$\dot{V}(x_p(t)) \leq 0 \quad \forall x_p(t) \in \mathcal{D}_{xp}$$

then $\mathcal{E}_{Vp}$ is a region of stability. Moreover, if

$$\dot{V}(x_p(t)) < 0 \quad \forall x_p(t) \in \mathcal{D}_{xp};\ x_p(t) \neq 0$$

then $\mathcal{E}_{Vp}$ is a region of asymptotic stability, i.e if $x_p(0) \in \mathcal{E}_{Vp}$, the trajectories $x_p(t) \to 0$ as $t \to \infty$

So, the existence of the function V provides a region of stability $\mathcal{E}_{Vp}$. The next corollary expresses this region of stability as a level set of V:

Corollary 1. *Suppose that V is a Lyapunov function of system* (11)*, i.e. it satisfies the conditions of . Let $\mathcal{E}_V = \{x \in \mathbb{R}^n : V(x) \leq c\}$, with $c > 0$. If $\mathcal{E}_V \subset \mathcal{D}_{xp}$, then it is an estimate of the region of attraction of system* (11) *i.e. all the solutions $x_p(t)$ of* (11) *with initial condition at $x_p(0) = x$ converge asymptotically to the origin as $t \to +\infty$.*

To end this section, it should be admitted that the exposition of technical details and formulations is somewhat unconventional in a bibliographic revision. Nevertheless, the information exposed in this section is important to this study and will appear again in the next chapter. Anyhow, in the following section we return to the revision specifically about saturation in SDS.

2.4 Sampled-data systems with input saturation

The earliest work concerned with a SDS subject to saturation that could be found is of DESOER; WING, 1961, where a control rule is proposed such that the output of a plant controlled by this rule tracks a reference signal in a finite number of samples. As in MOUSA; MILLER; MICHEL, 1986, another early precursor, this study is not flexible in the sense of providing a systematic routine for the synthesis of controlling rules. Although such a routine is unnecessary in their case because the rule is fixed, it is not surprising that this study belongs to a minority against the references of this revision that propose LMI based conditions for the systematic synthesis of controllers: the formulation of semidefinite optimization problems only began to be highly valued in the early 80's, when there was a realization that convex problems with LMI restrictions had the potential to be easily solved by computers (BOYD *et al.*, 1994).

In this sense and four decades later, LMI-based stability conditions for an aperiodic SDS were presented by FRIDMAN; SEURET; PIERRE RICHARD, 2004, as already cited. In their study, the aspect of saturation is specifically covered by a polytopic representation that was first used in HU; LIN; CHEN, 2002. Later, DAI *et al.*, 2010 have proposed the design of output feedback controllers with internal deadzone loops considering the generalized sector condition proposed in (GOMES DA SILVA Jr.; TARBOURIECH, 2005), and for a case with constant sampling period.

Not much later, following the timeline of the analysis techniques, SEURET; GOMES DA SILVA Jr., 2012 combined the then-recent looped functional approach with the generalized sector condition, to tackle the problem of stability of an aperiodic SDS with input saturation. In PALMEIRA, 2015, these functionals are employed in a case where the saturation affects not only the magnitude of the actuator, but its rate of change too.

More recently, in FIACCHINI; GOMES DA SILVA Jr., 2018, this problem is addressed considering an impulsive system framework, where the discrete-time dynamics of the closed-loop system is cast as a difference inclusion obtained from the partition of the intersampling interval, and the stability is attested by the verification of certain set invariance conditions through Lyapunov based conditions. Below, the impulsive system used in that paper is formalized with an extended state variable $x : \mathbb{R}^+ \to \mathbb{R}^n$ given by $x(t) = \left[x_p'(t)\; u'(t)\right]'$, and $n = n_p + m$.

$$\begin{cases} \dot{x}(t) = A_f x(t) & \forall t \notin \Theta; \\ x(t) = A_j x(t^-) + B_j \text{sat}(K_j x(t^-)) & \forall t \in \Theta; \quad t \neq 0 \\ x(0) = x_0 & t = 0 \end{cases} \tag{12}$$

The matrices A_f, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ and $K_j \in \mathbb{R}^{m\times n}$ are given as follows

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad A_j = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad K_j = \begin{bmatrix} K & 0 \end{bmatrix}$$

The classification of the stability conditions is very useful to compile these works and their methodologies. Inspired in the notation of BRIAT; SEURET, 2012, we may define *continuous*, *impulsive*, and *discrete* expressions of stability as $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{D}$, respectively:

$$\mathcal{C} \triangleq \dot{V}(x(t)) \qquad \forall t \notin \Theta \tag{13}$$

$$\mathcal{I} \triangleq V(x(t)) - V(x(t^-)) \qquad \forall t \in \Theta \tag{14}$$

$$\mathcal{D} \triangleq V(x(t_{k+1})) - V(x(t_k)) \qquad \forall t_k \in \Theta,\ k \in \mathbb{N},\ t_{k+1} > t_k \tag{15}$$

To simplify, these conditions are stated for a Lyapunov function V dependent only on $x(t)$, but they are still valid when it is a function of time, as is the case of the clock-dependent Lyapunov functions used in BRIAT, 2013, or even when they are a function of a function, as is the case of the functionals. Regarding again the paper of FIACCHINI; GOMES DA SILVA Jr., 2018, the conditions therein and that were mentioned earlier are expressed by $\mathcal{D} < 0$. Exploring this idea further, the following table summarizes some of the references cited before. In the approaches column, Hyb-CLF, Hyb-QF, TD-LKF, and TD-LF, stand for hybrid approach with clock-dependent Lyapunov functions, hybrid approach with quadratic Lyapunov function, time-delay approach with Lyapunov-Krasovskii functional, and time-delay approach with looped functional, respectively. In the formulation column, the optimization problems proposed by the studies are classified in semidefinite or sum-of-squares programmings.

Article	Conditions			Approach	Formulation	sat()
	$\mathcal{C} < 0$	$\mathcal{I} < 0$	$\mathcal{D} < 0$			
[1]	$\times$	$\times$		Hyb-CLF	SDP & SOSP	no
[2]			$\times$	Hyb-QF	SDP	yes
[3]	$\times$	$\times$		Hyb-CLF	SDP	no
[4]	$\times$			TD-LKF	SDP	yes
[5]	$\times$			TD-LKF	SDP	no
[6]			$\times$	TD-LF	SDP	yes
[7]			$\times$	TD-LF	SDP	no
[8]			$\times$	TD-LF	SDP	yes
[9]			$\times$	TD-LF	SDP	no

[1] (BRIAT, 2013)
[2] (FIACCHINI; GOMES DA SILVA Jr., 2018)
[3] (NAGHSHTABRIZI; HESPANHA; TEEL, 2008)
[4] (FRIDMAN; SEURET; PIERRE RICHARD, 2004)
[5] (FRIDMAN, 2010)
[6] (PALMEIRA, 2015)
[7] (SEURET, 2012)
[8] (SEURET; GOMES DA SILVA Jr., 2012)
[9] (BRIAT; SEURET, 2012)

It should be remarked that in this table of references, only the papers that have explicit results concerning saturation are marked with sat()=yes. Nevertheless, this classification is not final, as it should be possible to extend some of these results with varying degrees of difficulty. For example, it is stated in NAGHSHTABRIZI; HESPANHA; TEEL, 2008 that the stability and stabilization of sampled-data system with input saturation can be considered simply by following the steps of FRIDMAN; SEURET; PIERRE RICHARD, 2004.

2.5 Final comments

From the bibliography review presented in this chapter, it is understood how the stability analysis of a sampled-data system is approached from different angles, by different authors. Yet, to the best of the knowledge exposed in this chapter, no other study considering SDS with input saturation has ever been conducted in the hybrid systems framework, which implies that the contributions of this work are novel.

The contents of the next chapters is composed of these contributions, with the system (1) defined to represent a sampled-data system subject to saturation and aperiodic sampling. Based on the use of clock-dependent Lyapunov functions, the Chapter 4 presents an analysis method for the characterization of the region of stability associated with the equilibrium point of the system, and the Chapter 5 addresses the synthesis of a stabilizing controller.

3 PROBLEM FORMULATION AND PRELIMINARIES

3.1 Introduction

In this chapter, the SDS composed of a linear time-invariant plant and a digital controller subject to saturation is represented by a hybrid system, and the problems of stability emerging from this representation are stated. Approximating a networked control system, the plant state in the SDS is supposed to be sampled in variable intervals of time that are limited by upper and lower bounds. While the sampling intervals are supposed to be uncertain, only the deterministic aspects of the problem are regarded here (as is the case of every study surveyed by HETEL *et al.*, 2016), without a mention to the case where the sampling intervals are random variables given by a probability distribution.

With the aim at the problem of analysis of stability, the preliminary result of this chapter is developed from the theorem of clocked-dependent Lyapunov functions referenced from GOEBEL; SANFELICE; TEEL, 2009 in the bibliography. To handle the saturation of the controller, this nonlinearity is rewritten as a deadzone function and all of its effects are covered by the generalized sector condition referenced from TARBOURIECH *et al.*, 2011. The preliminary result provides conditions for the analysis of stability of the hybrid system considered.

3.2 Problem formulation

Consider the continuous-time plant described by the following linear time-invariant model:

$$\dot{x}_p(t) = Ax_p(t) + Bu(t)$$

where $x_p : \mathbb{R}^+ \to \mathbb{R}^{n_p}$ represents the state of the plant, $u : \mathbb{R}^+ \to \mathbb{R}^m$ represents its input, and $t \in \mathbb{R}^+$ is time. Matrices A and B have appropriate dimensions and are constant.

It is considered that the input signal is calculated from samples, as if coming from a digital controller, and is constrained in amplitude due to a saturation effect:

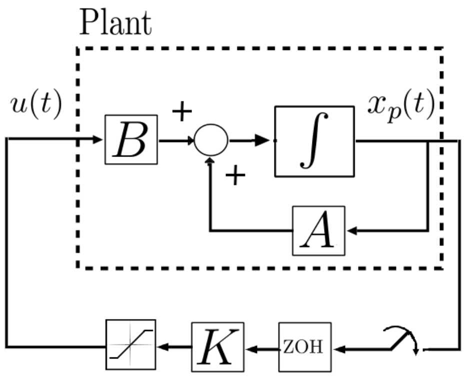
$$u(t) = \text{sat}(K(x_p(t_k))) \tag{16}$$

where sat : $\mathbb{R}^m \to \mathbb{R}^m$ is a vector valued saturation function that limits each element $v_{(i)}$ of an input vector $v \in \mathbb{R}^m$ to a range given by $[-u_{\text{sat}(i)},\ u_{\text{sat}(i)}]$, i.e.

$$\text{sat}(v)_{(i)} = \text{sign}(v_{(i)})\text{min}(|v_{(i)}|, u_{\text{sat}(i)}) \quad \forall i = 1, \cdots, m \tag{17}$$

The samples are hold by a zero order holder (ZOH), whose output is constant between updates. Consequently, the control signal $u(t)$ is kept constant between t_k and t_{k+1}.

Figure 2 – Closed-loop system



Source: adapted from PALMEIRA, 2015.

A block diagram of the control system just described can be visualized in Figure 2. The updates of the ZOH occur in a set of instants Θ, when the switch in the figure bypasses the value of $x_p(t)$. This set is described as follows:

$$\Theta = \{t = t_k : t_{k+1} = t_k + \delta_k, \quad \delta_k \in [\underline{\tau},\ \overline{\tau}]\, , \forall k \in \mathbb{N}^+\}$$

Since the sampling is not periodic, the intersampling time $\delta_k = t_{k+1} - t_k$ is not necessarily constant. It is assumed that it is limited between two bounds:

$$0 < \underline{\tau} \leq \delta_k \leq \overline{\tau}$$

The bounds $\underline{\tau}$ and $\overline{\tau}$ are supposed to be imposed by constraints on a networked control implementation. They can represent, for instance, network conditions that affect sampling rate e.g. a lag induced by the communication protocol.

In our approach, the representation for the closed-loop system is based on the hybrid systems framework. The hybrid system $\mathcal{H}(C, f, D, g)$ has an overall state variable $\eta(t) = [x'(t)\ \tau(t)]'$, with $x : \mathbb{R}^+ \to \mathbb{R}^n$ given by $x(t) = \left[x_p'(t)\ u'(t)\right]'$, and $n = n_p + m$. $\tau : \mathbb{R}^+ \to [0,\ \overline{\tau})$ is called the clock variable, and is given by $\tau(t) = t - t_k, \forall t \in [t_k, t_{k+1})$. The system $\mathcal{H}(C, f, D, g)$ represents the behaviour of the closed-loop system by the following equations:

$$\mathcal{H}\begin{cases}\dot{\eta}(t) = \begin{bmatrix}\dot{x}(t)\\ \dot{\tau}(t)\end{bmatrix} = f(\eta(t)) = \begin{bmatrix}A_f x(t)\\ 1\end{bmatrix}, & \forall\eta(t) \in C = \mathbb{R}^n \times [0,\ \overline{\tau})\\ \eta^+(t) = \begin{bmatrix}x^+(t)\\ \tau^+(t)\end{bmatrix} = g(\eta(t)) = \begin{bmatrix}\begin{pmatrix}A_j x(t)+\\ B_j \text{sat}(K_j x(t))\end{pmatrix}\\ 0\end{bmatrix}, & \forall\eta(t) \in D = \mathbb{R}^n \times [\underline{\tau},\ \overline{\tau}]\end{cases} \quad (18)$$

where A_f, $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$ and $K_j \in \mathbb{R}^{m\times n}$ are given as follows

$$A_f = \begin{bmatrix}A & B\\ 0 & 0\end{bmatrix}, \quad A_j = \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}, \quad B_j = \begin{bmatrix}0\\ I\end{bmatrix}, \quad K_j = \begin{bmatrix}K & 0\end{bmatrix}$$

Concerning system (18), the following problems of interest are stated:

P1. Given all parameters of system $\mathcal{H}$ (A_f, A_j, B_j, and K_j), the vector of saturation limits $u_{\text{sat}} \in \mathbb{R}^m$ and the sampling interval limits $\underline{\tau}$ and $\overline{\tau}$, estimate the region of attraction to the origin $x = 0$ of the closed loop sampled-data system.

P2. Given all parameters of system $\mathcal{H}$ (A_f, A_j, B_j, and K_j), the vector of saturation limits $u_{\text{sat}} \in \mathbb{R}^m$ and a set of admissible initial conditions $\mathcal{P} \in \mathbb{R}^{n_p}$, estimate the upper bound $\overline{\tau}$ such that, for all $x(0) \in \mathcal{P}$, the corresponding trajectories $x(t)$ converge asymptotically to the origin $x = 0$ of the closed loop sampled-data system.

P3. Given A_f, A_j and B_j, the vector of saturation limits $u_{\text{sat}} \in \mathbb{R}^m$ and the sampling interval limits $\underline{\tau}$ and $\overline{\tau}$, determine the feedback gain $K_j \in \mathbb{R}^{m\times n}$ that maximizes an estimate of the region of attraction to the origin $x = 0$ of the closed loop sampled-data system.

3.3 Preliminary results

In this section is developed a base theorem that addresses the problems **P1** and **P2**. The theorem provides sufficient conditions to certificate the stability of system (18), and, moreover, will serve as the basis for the theorems developed in the next chapter.

As mentioned in Section 2.3, the region of stability of a system is usually expressed as a level set of the Lyapunov function. The next theorem checks the stability of a system $\mathcal{H}(C, f, D, g)$ in the sense of Lyapunov with conditions to be satisfied by a Clock-dependent Lyapunov function.

Theorem 3. *(GOEBEL; SANFELICE; TEEL, 2009) Consider system $\mathcal{H}$ and a domain $\mathcal{D} \subset C \cup D \cup \{0\}$, where $\mathcal{D} = \mathcal{D}_x \times [0\ \overline{\tau}]$. Let c_1 and c_2 be positive constants and $V : \mathcal{D} \to \mathbb{R}^+$ be a function such that the following conditions are verified for any solution of* (18)*:*

$$c_1||x(t)||^2 \leq V(\eta(t)) \leq c_2||x(t)||^2 \quad \forall \eta(t) \in \mathcal{D}$$

$$\nabla V(\eta(t)) f(\eta(t)) < 0 \quad \forall \eta(t) \in \mathcal{D} \cap C \tag{19}$$

$$V(g(\eta(t))) - V(\eta(t)) < 0 \quad \forall \eta(t) \in \mathcal{D} \cap D \tag{20}$$

then, the equilibrium point $x = 0$ of system $\mathcal{H}$ is locally and asymptotically stable.

Since the SDS is subject to saturation, then its stability is possibly not defined over the entire space of its state. What is observed in this case is a reiteration of Section 2.3: in such system, the stability of an equilibrium point is not necessarily global. Therefore, for the continuation of this document, it is necessary an expression for the region of stability associated with the equilibrium point. The following corollary expresses this region.

Corollary 2. *Suppose that V verifies the conditions of Theorem 3. Let $\mathcal{E}_V = \{x \in \mathbb{R}^n : V(x, 0) \leq c\}$, for any $c > 0$. If $\mathcal{E}_V \subset \mathcal{D}_x$, then it is an estimate of the region of attraction of system $\mathcal{H}$ i.e. all the solutions $x(t)$ of* (18) *with initial condition $\eta(0) = [x' \ 0]'$ converge asymptotically to the origin as $t \to +\infty$.*

3.3.1 Saturation handling

Henceforth, conditions that are sufficient for the verification of those stated in Theorem 3 shall be developed. To this end, the conditions (19) and (20) will be adapted to cover the input saturation.

Consider that a *deadzone* function $\psi(v)$ is defined by:

$$\psi(v) = \text{sat}(v) - v$$

that is

$$\psi(v)_{(i)} = \text{sign}(v_{(i)}) \left(u_{\text{sat}(i)} - \max(|v_{(i)}|, u_{\text{sat}(i)}) \right) \tag{21}$$

The following lemma, denominated generalized sector condition, is useful to relax the conditions of Theorem 3 and obtain constructive conditions that are valid for a system subject to saturation.

Lemma 1. *(GOMES DA SILVA Jr.; TARBOURIECH, 2005) Consider $K_j, G_j \in \mathbb{R}^{m \times n}$ and define the set*

$$\mathcal{S} = \{x \in \mathbb{R}^n \ : \ |(K_{j(i)} - G_{j(i)})x| \leq u_{sat(i)}, \ i = 1, \cdots, m\} \tag{22}$$

If $x \in \mathcal{S}$, then the relation

$$\psi(K_j x)' T (\psi(K_j x) + G_j x) \leq 0 \tag{23}$$

is satisfied for any diagonal positive definite matrix $T \in \mathbb{R}^{m \times m}$.

Proof. Consider the three cases that follow:

1. $-u_{\text{sat}(i)} \leq K_{j(i)}x \leq u_{\text{sat}(i)}$. In this case, by equation (21), $\psi(K_{j(i)}x) = 0$ and then $\psi(K_{j(i)}x)'\tilde{T_{(i,i)}}(\psi(K_{j(i)}x) + G_jx) = 0$.

2. $K_{j(i)}x > u_{\text{sat}(i)}$. In this case, $\psi(K_{j(i)}x) = u_{\text{sat}(i)} - K_{j(i)}x$. If $x \in \mathcal{S}$, it follows that $(K_{j(i)} - G_{j(i)})x \leq u_{\text{sat}(i)}$. Hence, it follows that $\psi(K_{j(i)}x) + G_{j(i)}x = u_{\text{sat}(i)} - (K_{j(i)} - G_{j(i)})x \geq 0$ and, since in this case $\psi(K_{j(i)}x) < 0$, one gets $\psi(K_{j(i)}x)T_{(i,i)}(\psi(K_{j(i)}x) + G_jx) \leq 0$

3. $K_{j(i)}x < -u_{\text{sat}(i)}$. In this case, $\psi(K_{j(i)}x) = -u_{\text{sat}(i)} - K_{j(i)}x$. If $x \in \mathcal{S}$, it follows that $(K_{j(i)} - G_{j(i)})x \geq -u_{\text{sat}(i)}$. Hence, it follows that $\psi(K_{j(i)}x) + G_{j(i)}x = -u_{\text{sat}(i)} - (K_{j(i)} - G_{j(i)})x \leq 0$ and, since in this case $\psi(K_{j(i)}x) > 0$, one gets $\psi(K_{j(i)}x)T_{(i,i)}(\psi(K_{j(i)}x) + G_jx) \leq 0$

From these three cases, provided that $x \in \mathcal{S}$, we can conclude that $\psi(K_{j(i)}x)T_{(i,i)}(\psi(K_{j(i)}x) + G_jx) \leq 0,\ \forall i = 1, \cdots, m$, leading to the verification of (23) □

The next lemma can be seen as a link between the generalized sector condition from Lemma 1 and the next theorem, and will be recalled during the latter's proof. It was inspired in a procedure that can be found in BOYD, 2008.

Lemma 2. *Consider $T \in \mathbb{S}^m$ any diagonal positive definite matrix, and the matrices $G_j \in \mathbb{R}^{m\times n}$, $K_j \in \mathbb{R}^{m\times n}$, and $x, x^+ \in \mathbb{R}^n$, $\psi(K_jx) \in \mathbb{R}^m$.*

If (24) *is satisfied,*

$$\begin{bmatrix} x^+ \\ x \\ \psi(K_jx) \end{bmatrix}' \left(\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \star & M_{22} & M_{23} \\ \star & \star & M_{33} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & G_j'T \\ \star & \star & 2T \end{bmatrix} \right) \begin{bmatrix} x^+ \\ x \\ \psi(K_jx) \end{bmatrix} < 0 \qquad (24)$$

then, provided that $x \in \mathcal{S}$, (25) *is satisfied as well.*

$$\begin{bmatrix} x^+ \\ x \\ \psi(K_jx) \end{bmatrix}' \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \star & M_{22} & M_{23} \\ \star & \star & M_{33} \end{bmatrix} \begin{bmatrix} x^+ \\ x \\ \psi(K_jx) \end{bmatrix} < 0 \qquad (25)$$

Proof. Defining

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \star & M_{22} & M_{23} \\ \star & \star & M_{33} \end{bmatrix}, \qquad y = \begin{bmatrix} x^+ \\ x \\ \psi(K_jx) \end{bmatrix}$$

then (24) is equivalent to:

$$y'My - 2\psi(K_jx)T(\psi(K_jx) + G_jx) < 0$$

which leads to

$$y'My < 2\psi(K_jx)T(\psi(K_jx) + G_jx) \tag{26}$$

Applying Lemma 1, the following proposition holds

$$x \in \mathcal{S} \Rightarrow \psi(K_jx)T(\psi(K_jx) + G_jx) \leq 0$$

Since (26) is equivalent to (24), it is evident that if (24) is verified and $x \in \mathcal{S}$, then $y'My < 0$ and (25) is verified as well, which concludes the proof. □

In order to apply Lemma 2 in the stability analysis, the saturation function in (18) must be substituted by the deadzone function. The substitution results in

$$g(\eta(t)) = \begin{bmatrix} \mathbb{A}_jx(t) + B_j\psi(K_jx(t)) \\ 0 \end{bmatrix} \quad \forall \eta(t) \in D \tag{27}$$

with $\mathbb{A}_j = A_j + B_jK_j$.

The forthcoming results are applied to system (18) considering $g(\eta(t))$ as in (27).

3.3.2 Quadratic Lyapunov Function

As already introduced, the following theorem is a preliminary result for the development of the subsequent theorems. This theorem, based on quadratic Lyapunov function candidates to Theorem 3, provides sufficient stability conditions for the system (18), and is the first contribution of this study :

Theorem 4. *Consider the hybrid system $\mathcal{H}$ given in* (18)*. If there exist a matrix function $P : [0,\ \overline{\tau}] \rightarrow \mathbb{S}^n$, and matrices $G_j \in \mathbb{R}^{m\times n}$, $N \in \mathbb{R}^{2n+m\times n}$ and a diagonal matrix $T \in \mathbb{R}^{m\times m}$ that satisfy the following inequalities*

$$P(\tau) > 0 \quad \forall \tau \in [0,\ \overline{\tau}] \tag{28}$$

$$A_f'P(\tau) + P(\tau)A_f + \dot{P}(\tau) < 0 \quad \forall \tau \in [0,\ \overline{\tau}) \tag{29}$$

$$\Lambda(\tau) + N\Gamma + \Gamma'N' < 0 \quad \forall \tau \in [\underline{\tau},\ \overline{\tau}] \tag{30}$$

$$\begin{bmatrix} P(\tau) & (K_{j(i)} - G_{j(i)})' \\ \star & u_{0(i)}^2 \end{bmatrix} \geq 0 \quad \begin{matrix} \forall \tau \in [\underline{\tau},\ \overline{\tau}] \\ i = 1, \cdots, m \end{matrix} \tag{31}$$

with

$$\begin{aligned} \Lambda(\tau) &= \begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & -G_j'T \\ \star & \star & -2T \end{bmatrix} \\ \Gamma &= \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix} \\ \mathbb{A}_j &= A_j + B_jK_j \end{aligned} \tag{32}$$

then, considering a quadratic Lyapunov function $V(\eta(t)) = V(x(t), \tau(t)) = x'(t)P(\tau(t))x(t)$*, all initial conditions of system* (18) *such that* $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n \ : \ V(x,0) = x'P(0)x \leq 1\}$ *result in solutions* $x(t)$ *that converge asymptotically to the origin as* $t \to +\infty$.

Proof. Consider $x(t) \triangleq x$ and $\tau(t) \triangleq \tau$ and some solution of system (18) given by $\eta(t) = [x'(t) \ \tau(t)]$. From the following identity

$$\nabla V(x,\tau) f(\eta(t)) = x' \left(A_f' P(\tau) + P(\tau) A_f + \dot{P}(\tau) \right) x \tag{33}$$

it is deduced that the verification of (29) leads to $\nabla V(x,\tau) f(\eta(t)) < 0 \ \forall x \in \mathbb{R}^n, \ \forall \tau \in [0, \ \overline{\tau})$. Taking into account that $C = \mathbb{R}^n \times [0, \ \overline{\tau})$, it follows that

$$\nabla V(x,\tau) f(\eta(t)) < 0 \ \forall x \in \mathbb{R}^n, \ \forall \tau \in [0, \ \overline{\tau}) \ \Leftrightarrow \ \nabla V(\eta(t)) f(\eta(t)) < 0 \quad \forall \eta(t) \in C$$

In other words, the condition (29) is equivalent to condition (19) of Theorem 3.

On the other hand, the condition (20) of Theorem 1 can be cast as follows

$$\begin{gathered}\begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix}' \begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & 0 \\ \star & \star & 0 \end{bmatrix} \begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix} < 0 \\ \forall \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix} \begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix} = 0, \ \forall \tau \in [\underline{\tau}, \ \overline{\tau}] \end{gathered} \tag{34}$$

By applying Lemma 2 with the following instances

$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ \star & M_{22} & M_{23} \\ \star & \star & M_{33} \end{bmatrix} = \begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & 0 \\ \star & \star & 0 \end{bmatrix}$$

it is ensured that, provided $x \in \mathcal{S}$, if

$$\begin{gathered}\begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix}' \left(\begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & 0 \\ \star & \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & G_j' T \\ \star & \star & 2T \end{bmatrix} \right) \begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix} \leq 0 \\ \forall \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix} \begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix} = 0, \ \forall \tau \in [\underline{\tau}, \ \overline{\tau}] \end{gathered} \tag{35}$$

then (34) is verified. The Finsler's Lemma (Appendix 7.1, Lemma 4) is now applied with the following instances:

$$\Gamma = \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & 0 \\ \star & \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ \star & 0 & G_j'T \\ \star & \star & 2T \end{bmatrix}$$

$$y = \begin{bmatrix} x^+ \\ x \\ \psi(K_j x) \end{bmatrix}$$

By the Statement 1 and Statement 4 of Lemma 4, it is concluded that (30) is equivalent to (35).

With this information, we know that $((x \in \mathcal{S})$ & (30)) $\Rightarrow$ (34) $\Leftrightarrow$ (20). Therefore, the only remaining step in order to obtain the desired relation of sufficiency (30) $\Rightarrow$ (20) is to ensure that $x \in \mathcal{S}\ \forall \tau \in [\underline{\tau},\ \overline{\tau}]$. To this end, first consider that the initial state of system (18) is such that $x(0) \in \mathcal{E}_V$, as stated by the Theorem 4 currently being proved. With this consideration, we have the relation that follows:

$$1 \geq x'(0)P(0)x(0) \tag{36}$$

Then, recall the identity (33) respective to condition (19) and notice the extra term in the equation below:

$$\nabla V(x,\tau)f(\eta(t)) = x'\left(A_f'P(\tau) + P(\tau)A_f + \dot{P}(\tau)\right)x = \frac{d(x'P(\tau)x)}{dt} \tag{37}$$

From (37) we have that if (29) is satisfied, then

$$\frac{d(x'P(\tau)x)}{dt} < 0 \qquad \forall x \in \mathbb{R}^n, \quad \forall \tau \in [0,\ \overline{\tau})$$

Consider now two scalars $\tau_2 > \tau_1$. Despite the negative derivative above holding for all $\tau \in [0,\ \overline{\tau})$, at this point we can only be sure that $x'P(\tau_1)x > x'P(\tau_2)x$ if $\tau_1,\ \tau_2 \in [0, \underline{\tau}]$. The reason behind this fact is that the relation (30) $\Rightarrow$ (20) has not been validated yet by this proof, and thus, as far as is known, jumps of the system resulting in discontinuities such that $x'P(\tau_1)x < x'P(\tau_2)x$ when $\tau_1,\ \tau_2 \in [\underline{\tau},\ \overline{\tau}]$ could occur. With this information, suppose that it is known that the jump occurs at $\tau = \hat{\tau} \in [\underline{\tau},\ \overline{\tau}]$. In this case, the relation (36) can be extended to

$$1 \geq x'P(\tau)x \qquad \forall x \in \mathbb{R}^n, \quad \forall \tau \in [0, \hat{\tau}] \tag{38}$$

and then multiplied by $u^2_{\text{sat}(i)}$ to result in

$$u^2_{\text{sat}(i)} \geq u^2_{\text{sat}(i)}x'P(\tau)x \qquad \forall x \in \mathbb{R}^n, \quad \forall \tau \in [0, \hat{\tau}] \tag{39}$$

On the other hand, observe that if (31) is verified, then, from Schur complement (Appendix 7.3, Lemma 5), it follows that:

$$P(\tau) - (K_{(i)} - G_{(i)})'u^{-2}_{\text{sat}(i)}(K_{(i)} - G_{(i)}) \geq 0 \quad \begin{array}{l}\forall \tau \in [\underline{\tau},\ \overline{\tau}] \\ i = 1, \cdots, m\end{array} \tag{40}$$

After respectively pre and post multiplying (40) by x' and x, the following equivalent condition is obtained

$$u_{\text{sat}(i)}^2 x'P(\tau)x \geq x'(K_{(i)} - G_{(i)})'(K_{(i)} - G_{(i)})x \qquad \forall x \in \mathbb{R}^n,\ \forall \tau \in [\underline{\tau},\ \overline{\tau}] \quad i = 1, \cdots, m \tag{41}$$

From (39) and (41), we conclude that if (29) and (31) are satisfied, then

$$u_{\text{sat}(i)}^2 \geq u_{\text{sat}(i)}^2 x'P(\tau)x \geq x'(t)(K_{(i)}-G_{(i)})'(K_{(i)}-G_{(i)})x(t) \qquad \forall x \in \mathbb{R}^n,\ \forall \tau \in [\underline{\tau}, \hat{\tau}] \quad i = 1, \cdots, m$$

which implies that:

$$u_{\text{sat}(i)} \geq |(K_{(i)} - G_{(i)})x| \qquad \forall x \in \mathbb{R}^n,\ \forall \tau \in [\underline{\tau}, \hat{\tau}] \quad i = 1, \cdots, m \tag{42}$$

In other words, (42) implies that $x \in \mathcal{S}\ \forall \tau \in [\underline{\tau}, \hat{\tau}]$. Observe that now the relation (30) $\Rightarrow$ (20) is proven to be valid at the jump instant $\hat{\tau}$, and thus it is deduced that $1 \geq x'P(\tau)x\ \forall \tau \in [0,\ \overline{\tau}]$, which implies, through the same steps that led from (38) to (42), that $x \in \mathcal{S}\ \forall \tau \in [\underline{\tau},\ \overline{\tau}]$ and therefore that (30) $\Rightarrow$ (34) is valid.

The final conclusion is that if the inequalities of Theorem 4 are satisfied, then the inequalities of Theorem 3 are satisfied as well, which finishes the proof. □

3.4 Final comments

In this chapter, a formal representation of a linear SDS subject to input saturation and aperiodic sampling was presented in the framework of hybrid systems. From the hybrid representation, the problems of interest regarding stability and its analysis and synthesis (via stabilizing controllers) were formalized. Lyapunov based stability conditions were thus developed for the analysis of this class of system with the assumption that the Lyapunov function candidate has a form that is quadratic in the state of the system. The conditions are presented in Theorem 4, which is a first contribution of this work.

In Theorem 4, the candidate is assumed to be quadratic, but no assumption about its dependency on the clock variable is made. The specific type of the candidate function is a matter that will be expounded in the next chapter.

4 STABILITY ANALYSIS

4.1 Introduction

In this chapter, the problems **P1** and **P2** are tackled, which concern the stability analysis of the feedback control system in Figure 2. Recapitulating the last chapter, the sampled-data system in that figure has been represented by a hybrid system, and some conditions were developed to sufficiently ensure its stability. The results presented here, which are based on these conditions, are the result of additional hypotheses being assumed (in the same manner that Theorem 3 has resulted in Theorem 4 after the assumption of a quadratic form Lyapunov function) and also of the application of instrumental lemmas.

In practice, it would be difficult to "manually" find a feasible set of values for the constraints of Theorem 4, without the help of a solving algorithm. Moreover, bearing in mind that each solution is associated to a different estimate of the region of attraction, some solutions reveal more information about the real region of attraction than others, and so are, simply put, more valuable. Hence, it is important to formulate the constraints of Theorem 4 in semidefinite or sum-of-squares programmings, as this formulation renders well-posed optimization problems concerning these two points.

It is usual to formulate the type of problems addressed in this chapter in the semidefinite programming, with the constraints as linear matrix inequalities (LMI) (see VANDENBERGHE; BOYD, 1996 for a review of the matter and BOYD *et al.*, 1994 for a survey showing its frequent application in control theory). In the next section, our intent is to arrive at this formulation through a hypothesis that $P(\tau)$, that defines the Lyapunov function of Theorem (4), has affine dependency on τ. This hypothesis is assumed because, unfortunately, the LMIs that could be obtained from equations (29), (30) and (31) would be parameterized (also known as *robust linear matrix inequalities* (OISHI; FUJIOKA, 2010)) by τ in $[0,\ \overline{\tau}]$ and $[\underline{\tau},\ \overline{\tau}]$, leading to infinite LMIs and making it impossible to check the existence of the candidate Lyapunov function via semidefinite programming. The next section begins stating explicitly this affine dependency, which has been already used in previous studies that will be cited.

After the next section, the Lyapunov candidate function is allowed to have a polynomial dependence on τ, which is more general than the affine one. It is not the intent of this chapter to propose conditions based on parameterized/robust LMIs, and for this cause the theorems of the Section 4.3 propose conditions for the the sum-of-squares programming. It is also not the intent of this chapter to provide information in excess to what is necessary for the exposition of the results, so the reader interested in a more in-depth explanation about the issue of changing from SDP to SOSP should refer to Appendix 7.2, particularly to Section 7.2.1.

4.2 Affine clock-dependent Lyapunov function

We introduce a particular form of quadratic Lyapunov functions:

$$V(\eta(t)) = V(x(t), \tau(t)) = x'(t)(P_0 + \tau(t)P_1)x(t) \tag{43}$$

That form has been used previously in BOYARSKI; SHAKED, 2009, ALLERHAND; SHAKED, 2011 and HU *et al.*, 2003 in virtue of the affine dependence on the clock $\tau(t)$. In particular, through convexity arguments, this form allows the conditions of Theorem 4 to be cast in terms of LMIs. This result is exposed in the next theorem:

Theorem 5. *If there exist matrices $P_0 \in \mathbb{S}^n$, $P_1 \in \mathbb{S}^n$, $G_j \in \mathbb{R}^{m\times n}$, $N \in \mathbb{R}^{2n+m\times n}$ and a diagonal matrix $T \in \mathbb{R}^{m\times m}$ that satisfy the following inequalities:*

$$P_0 > 0 \tag{44}$$

$$P_0 + \overline{\tau}P_1 > 0 \tag{45}$$

$$A_f'P_0 + P_0A_f + P_1 < 0 \tag{46}$$

$$A_f'(P_0 + \overline{\tau}P_1) + (P_0 + \overline{\tau}P_1)A_f + P_1 < 0 \tag{47}$$

$$\Lambda_1 + N\Gamma + \Gamma'N' < 0 \tag{48}$$

$$\Lambda_2 + N\Gamma + \Gamma'N' < 0 \tag{49}$$

$$\begin{bmatrix} (P_0 + \overline{\tau}P_1) & (K_{j(i)} - G_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0 \tag{50}$$

$$\begin{bmatrix} (P_0 + \underline{\tau}P_1) & (K_{j(i)} - G_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0, \quad i = 1, \cdots, m \tag{51}$$

with

$$\begin{aligned}
\Lambda_1 &= \begin{bmatrix} P_0 & 0 & 0 \\ \star & -P_0 - \underline{\tau} P_1 & -G_j' T \\ \star & \star & -2T \end{bmatrix} \\
\Lambda_2 &= \begin{bmatrix} P_0 & 0 & 0 \\ \star & -P_0 - \overline{\tau} P_1 & -G_j' T \\ \star & \star & -2T \end{bmatrix} \\
\Gamma &= \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix} \\
\mathbb{A}_j &= A_j + B_j K_j
\end{aligned} \tag{52}$$

then, considering the quadratic Lyapunov function $V(x(t), \tau(t))$ *given by* (43)*, all initial conditions of system* (18) *such that* $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n \ : \ V(x, 0) = x' P_0 x \leq 1\}$ *result in solutions* $x(t)$ *that converge asymptotically to the origin as* $t \to +\infty$*.*

Proof. Note that the pairs of constraints are related to the ones in Theorem 4. The inequalities are related by the logical propositions that follow:

$$\begin{aligned}
&(44) \text{ and } (45) \Rightarrow (28) \quad ; \quad (46) \text{ and } (47) \Rightarrow (29) \\
&(48) \text{ and } (49) \Rightarrow (30) \quad ; \quad (50) \text{ and } (51) \Rightarrow (31)
\end{aligned} \tag{53}$$

We start the proof with (44) and (45). Since we know that $P(\tau)$ is determined by Equation (43), if both are true, then

$$P(\tau) = P_0 + \tau P_1 = \left(\frac{\overline{\tau} - \tau}{\overline{\tau}}\right) P_0 + \frac{\tau}{\overline{\tau}} (P_0 + \overline{\tau} P_1) > 0 \quad \forall \tau \in [0,\ \overline{\tau}]$$

since

$$P_0 > 0 \text{ and } P_0 + \overline{\tau} P_1 > 0$$

Noting that $\dot{P}(\tau) = \dot{\tau} P_1 = P_1$, the same convexity argument can be used to prove that the other pairs of constraints ensure (29), (30) and (31) in their respective domains of τ:

$$\begin{aligned}
&A_f' P(\tau) + P(\tau) A_f + \dot{P}(\tau) = A_f'(P_0 + \tau P_1) + (P_0 + \tau P_1) A_f + P_1 = \\
&\left(\frac{\overline{\tau} - \tau}{\overline{\tau}}\right) \left(A_f' P_0 + P_0 A_f + P_1\right) + \frac{\tau}{\overline{\tau}} (A_f'(P_0 + \overline{\tau} P_1) + (P_0 + \overline{\tau} P_1) A_f + P_1) < 0 \\
&\qquad \forall \tau \in [0,\ \overline{\tau}]
\end{aligned}$$

$$\begin{aligned}
&\Lambda(\tau) + N\Gamma + \Gamma' N' = \\
&\left(\frac{\overline{\tau} - \tau}{\overline{\tau} - \underline{\tau}}\right) (\Lambda_1 + N\Gamma + \Gamma' N') + \left(\frac{\tau - \underline{\tau}}{\overline{\tau} - \underline{\tau}}\right) (\Lambda_2 + N\Gamma + \Gamma' N') < 0 \\
&\qquad \forall \tau \in [\underline{\tau},\ \overline{\tau}]
\end{aligned}$$

$$\begin{bmatrix} P(\tau) & (K_{j(i)} - G_{j(i)})' \\ \star & u_{0(i)} \end{bmatrix} =$$

$$\left(\frac{\overline{\tau} - \tau}{\overline{\tau} - \underline{\tau}}\right) \begin{bmatrix} P_0 + \underline{\tau} P_1 & (K_{j(i)} - G_{j(i)})' \\ \star & u_{0(i)} \end{bmatrix} + \left(\frac{\tau - \underline{\tau}}{\overline{\tau} - \underline{\tau}}\right) \begin{bmatrix} P_0 + \overline{\tau} P_1 & (K_{j(i)} - G_{j(i)})' \\ \star & u_{0(i)} \end{bmatrix} \geq 0$$

$$\forall \tau \in [\underline{\tau},\ \overline{\tau}]$$

□

As stated in the introduction of this chapter, this result can be seen as a way of casting the conditions in Theorem 4 as LMIs. It should be remarked that, actually, a final step of fixing either T or G_j to constant values is necessary in order to cast inequalities (48) and (49) as LMIs. Notwithstanding, the problem **P1** can be solved after that step is taken. Moreover, various methods are known to efficiently solve semidefinite programing problems with guaranteed convergency to the optimal solution, and consequently the solution is computable in a systematic way.

It is a fact that there always exists a certain degree of conservatism associated to the analysis of this problem. Two questions that naturally arise at this point are: what are the particularities that could be generalized with the intent of reducing the conservatism of this analysis? and what are the necessary steps for this extension to remain solvable in a systematic manner? The next section focus on these questions by eliminating our hypothesis of affine dependecy on τ, and substituting it by a polynomial, more general dependency.

4.3 Clock-dependent Lyapunov function with polynomial dependence on time

We begin this section by defining that a univariate matrix polynomial (UMP) of degree d is a finite linear combination of monomials. The next expression describes $P(\tau)$ as a UMP:

$$P(\tau) = \sum_{i=0}^{d} P_i \tau^i = P_0 + \tau P_1 + \cdots + \tau^d P_d \tag{54}$$

As stated in the introduction of this chapter, the attempt to write stability conditions for problem **P1** with $d > 1$ in semidefinite programming would require infinite LMI constraints to effectively enforce the positive definiteness of $P(\tau)$, each enforcing it for one value of τ inside a desired interval $[0,\ \overline{\tau}]$. This limitation can be overcome by the *sum of squares programming* (SOSP).

The sum of squares decomposition of the UMP in (54) is given by

$$P(\tau) = H'(\tau) H(\tau) \tag{55}$$

Specifically in our context, where $P : \mathbb{R} \to \mathbb{S}^n$, every SOS decomposition will result in a $H : \mathbb{R} \to \mathbb{R}^{s\times n}$ that is a UMP. Generally speaking, thought, if we considered $P : \mathbb{R}^z \to \mathbb{S}^h$, then $H : \mathbb{R}^z \to \mathbb{R}^{s\times h}$ would be classified as a multivariate matrix polynomial. This general concept of a sum of squares is not essential here, but is explored in the Appendix 7.2 for the reader interested in a background.

When a polynomial can be decomposed into a sum of squares, it is abbreviated that the polynomial *is* a sum of squares or, yet, that it *is SOS*. The proposition that follows exposes a property of SOS polynomials that is explored in the next theorem.

Proposition 1. *(PARRILO, 2000)*
If a UMP given by (54) *is SOS, then it is semidefinite positive* $\forall \tau$.

The next theorem casts the inequalities of Theorem 4 as constraints in the SOSP.

Theorem 6. *If there exist matrix polynomials* $P : \mathbb{R} \to \mathbb{S}^n$*,* Q_1*,* Q_2*,* Q_3 *and* $Q_4 : \mathbb{R} \to \mathbb{S}^n$*, matrices* $G_j \in \mathbb{R}^{m\times n}$*,* $N \in \mathbb{R}^{2n+m\times n}$*, a diagonal matrix* $T \in \mathbb{R}^{m\times m}$ *and a scalar* $\gamma > 0$ *that satisfy the following conditions:*

$$Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau) \text{ are SOS} \tag{56}$$

$$P(\tau) - \gamma I - Q_1(\tau)\tau(\overline{\tau} - \tau) \text{ is SOS} \tag{57}$$

$$-(A_f' P(\tau) + P(\tau) A_f + \dot{P}(\tau)) - \gamma I - Q_2(\tau)\tau(\overline{\tau} - \tau) \text{ is SOS} \tag{58}$$

$$-(\Lambda(\tau) + N\Gamma + \Gamma' N') - \gamma I - Q_3(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \text{ is SOS} \tag{59}$$

$$\begin{bmatrix} P(\tau) & (K_{j(i)} - G_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} - Q_4(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \text{ is SOS} \quad i = 1, \cdots, m \tag{60}$$

with

$$\begin{aligned} \Lambda(\tau) &= \begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & -G_j' T \\ \star & \star & -2T \end{bmatrix} \\ \Gamma &= \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix} \\ \mathbb{A}_j &= A_j + B_j K_j \end{aligned} \tag{61}$$

then, considering a quadratic Lyapunov function $V(\eta(t)) = V(x(t), \tau(t)) = x'(t)P(\tau(t))x(t)$ *with* $P : \mathbb{R} \to \mathbb{S}^n$ *given by* (54)*, all initial conditions of system* (18) *such that* $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n \ : \ V(x, 0) = x'P(0)x \leq 1\}$ *result in solutions* $x(t)$ *that converge asymptotically to the origin as* $t \to +\infty$*.*

Proof. Note that the inequalities of this theorem are sufficient conditions for the ones in Theorem 4:

$$\begin{array}{ccc} (57) \Rightarrow (28) & ; & (58) \Rightarrow (29) \\ (59) \Rightarrow (30) & ; & (60) \Rightarrow (31) \end{array} \tag{62}$$

Before proving each of the propositions above, observe that if (56) is verified, then, by Proposition 1,

$$Q_1(\tau) \geq 0,\ Q_2(\tau) \geq 0,\ Q_3(\tau) \geq 0,\ Q_4(\tau) \geq 0 \quad \forall \tau$$

Since $Q_1(\tau), \cdots, Q_4(\tau)$ are semidefinite positive $\forall \tau$, they can be used as slack variables[1] to ensure the definiteness of the terms that are not slack variables. To illustrate this concept, consider the following inequality:

$$P(\tau) - Q_1(\tau) \geq 0 \quad \forall \tau \tag{63}$$

Observe that the inequality above is sufficient to ensure that $P(\tau) \geq 0\ \forall \tau$. We want to validate the proposition (57) $\Rightarrow$ (28) by showing that (57) is sufficient for $P(\tau) > 0\ \forall \tau \in [0,\ \overline{\tau}]$. The valid domain of (63) can be reduced to $[0,\ \overline{\tau}]$ with the following expression:

$$P(\tau) - Q_1(\tau)\tau(\overline{\tau} - \tau) \geq 0 \quad \forall \tau$$

This inequality is sufficient to ensure that $P(\tau) \geq 0\ \forall \tau \in [0,\ \overline{\tau}]$. The next step is to ensure a strict inequality, with the term $\gamma > 0$:

$$P(\tau) - \gamma I - Q_1(\tau)\tau(\overline{\tau} - \tau) \geq 0 \quad \forall \tau \tag{64}$$

This inequality is always verified when (57) is satisfied, and, moreover, it is sufficient to ensure that $P(\tau) > 0 \quad \forall \tau \in [0,\ \overline{\tau}]$. The validity of (57) $\Rightarrow$ (28) is proven.

Likewise, the following inequalities

$$-(A_f' P(\tau) + P(\tau)A_f + \dot{P}(\tau)) - \gamma I - Q_2(\tau)\tau(\overline{\tau} - \tau) \geq 0 \quad \forall \tau$$

$$-(\Lambda(\tau) + N\Gamma + \Gamma' N') - \gamma I - Q_3(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \geq 0 \quad \forall \tau$$

$$\begin{bmatrix} P(\tau) & (K_{j(i)} - G_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} - \ Q_4(\tau)(\tau - \underline{\tau})(\overline{\tau} - \tau) \geq 0 \quad \forall \tau \quad i = 1, \cdots, m$$

are always verified when (58), (59) and (60) are satisfied, respectively. They also are, for the same reason that (64) $\Rightarrow$ (28), sufficient for the verification of (29), (30) and (31). Thus, the validity of the remaining propositions is proven, and the proof is complete. □

[1] Actually, $Q_1, ..., Q_4$ are not slack variables in the *strict* sense of the word, but we name them so as they ensure the definiteness of the other terms in their respective SOS constraints, in a comparable manner that a slack variable, strictly speaking, ensures the definiteness of the other terms in an equality constraint.

It is worth remarking that the region $\mathcal{E}_V$ is not a set containing only the trajectories of the plant state, as it is not defined in the same space of the plant state $x_p(t) \in \mathbb{R}^{n_p}$. It is, actually, defined in the space of $x(t) = [x_p'(t)\ u'(t)] \in \mathbb{R}^{n_p+m}$, and contains the trajectory of the control signal as well.

Our interest, in discordance with this fact, is to get an estimate of the region of convergence in terms of x_p, i.e. to define a set of admissible initial states $x_p(0)$ such that the trajectories that start in it always converge to the origin. Fortunately it is possible to make a projection of this region in $\mathbb{R}^{n_p}$, that is given by the set $\mathcal{E}_{Vp}(P) = \{x_p \in \mathbb{R}^{n_p} : [x_p'\ \text{sat}(Kx_p)']P[x_p'\ \text{sat}(Kx_p)']' \leq 1\}$ (see Remark 3 of FIACCHINI; GOMES DA SILVA Jr., 2018).

As proposed by GOMES DA SILVA Jr.; TARBOURIECH, 1999 for a saturation map $\text{sat}(Kx_p)$, the space $\mathbb{R}^{n_p}$ can be divided in *saturation regions* $S_1, ..., S_{3^m}$. The regions are described by polyhedral sets,

$$S_j(R_j, d_j) = \{x \in \mathbb{R}_p^n\ :\ R_j x \leq d_j,\ j = 1, \cdots, 3^m\}$$

where R_j and d_j are given by

$$R_1 = \begin{bmatrix} K \\ -K \end{bmatrix}$$

$$d_1 = \begin{bmatrix} u_0 \\ u_0 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} -K_{(1)} \\ K_{(2)} \\ \vdots \\ K_{(m)} \\ -K_{(2)} \\ \vdots \\ -K_{(m)} \end{bmatrix} \quad R_3 = \begin{bmatrix} -K_{(1)} \\ -K_{(2)} \\ K_{(3)} \\ \vdots \\ K_{(m)} \\ -K_{(3)} \\ \vdots \\ -K_{(m)} \end{bmatrix} \quad R_4 = \begin{bmatrix} K_{(1)} \\ -K_{(2)} \\ K_{(3)} \\ \vdots \\ K_{(m)} \\ -K_{(1)} \\ -K_{(3)} \\ \vdots \\ -K_{(m)} \end{bmatrix} \quad R_5 = \begin{bmatrix} -K_{(2)} \\ K_{(3)} \\ \vdots \\ K_{(m)} \\ K_{(1)} \\ -K_{(3)} \\ \vdots \\ -K_{(m)} \end{bmatrix} \ ..\ R_{3^m} = \begin{bmatrix} K_{(1)} \\ \vdots \\ K_{(m-1)} \\ -K_{(m)} \\ -K_{(1)} \\ \vdots \\ -K_{(m-1)} \end{bmatrix}$$

$$d_2 = \begin{bmatrix} -u_{(1)} \\ u_{(2)} \\ \vdots \\ u_{(m)} \\ u_{(2)} \\ \vdots \\ u_{(m)} \end{bmatrix} \; d_3 = \begin{bmatrix} -u_{(1)} \\ -u_{(2)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \end{bmatrix} \; d_4 = \begin{bmatrix} u_{(1)} \\ -u_{(2)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \\ u_{(1)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \end{bmatrix} \; d_5 = \begin{bmatrix} -u_{(2)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \\ -u_{(1)} \\ u_{(3)} \\ \vdots \\ u_{(m)} \end{bmatrix} \; ... \; d_{3^m} = \begin{bmatrix} u_{(1)} \\ \vdots \\ u_{(m-1)} \\ -u_{(m)} \\ u_{(1)} \\ \vdots \\ u_{(m-1)} \end{bmatrix}$$

For the x_p inside some region S_j, x is given by a continuous function $x_{S\_j} : \mathbb{R}^n \to \mathbb{R}^n$, as expressed by (65) and the equations above it. Since x is a function of x_p that has continuous derivative only when $x_p \notin \partial S_j$, i.e. is continuous *by parts*, the level set given by $\mathcal{E}_{Vp}(P)$ and defined by the relation $x'Px \leq 1$ is quadratic by parts as well, or, as named in the literature, *piece-wise quadratic*. The following equations define the continuous functions x_{S_j}, for $j = 1, \cdots, 3^m$:

$$x_{S\_1} = \begin{bmatrix} x_p \\ u_1(x_p) \end{bmatrix}, \quad u_1(x_p) = Kx_p$$

$$x_{S\_j} = \begin{bmatrix} x_p \\ u_j(x_p) \end{bmatrix}, \quad j = 1, 2, ..., 3^m$$

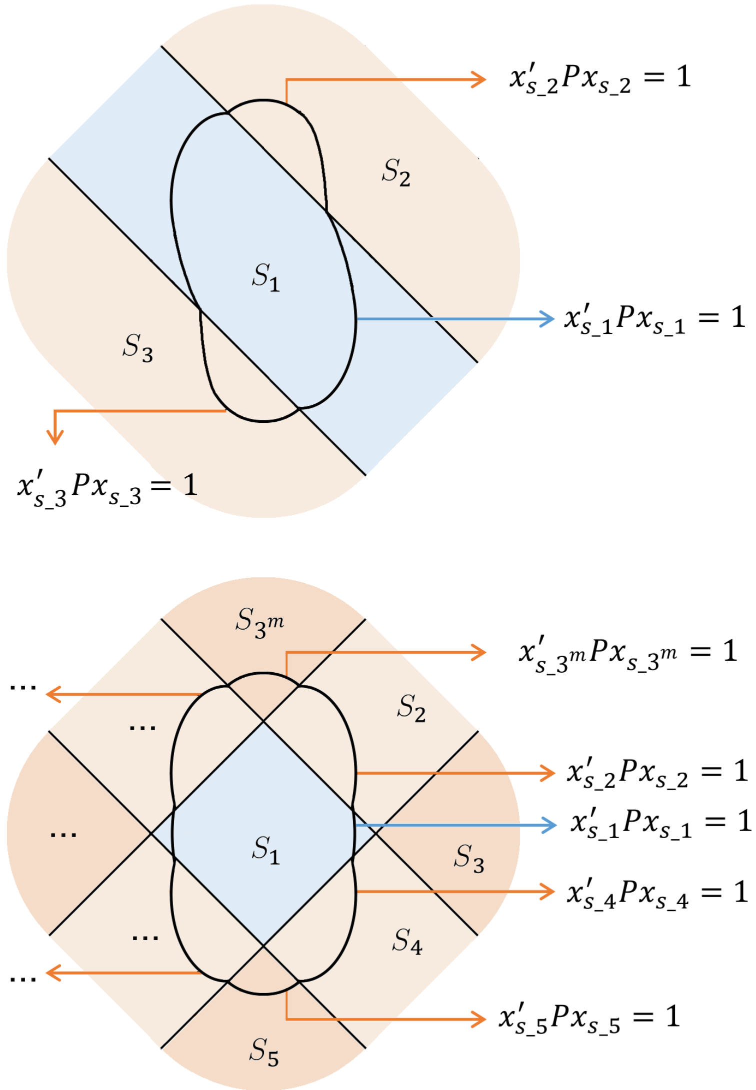
$$u_2(x_p) = \begin{bmatrix} u_{\text{sat}(1)} \\ K_{p(2)}x_p \\ K_{p(3)}x_p \\ \vdots \\ K_{p(m)}x_p \end{bmatrix} \; u_3(x_p) = \begin{bmatrix} u_{\text{sat}(1)} \\ u_{\text{sat}(2)} \\ K_{p(3)}x_p \\ \vdots \\ K_{p(m)}x_p \end{bmatrix} \; u_4(x_p) = \begin{bmatrix} K_{p(1)}x_p \\ u_{\text{sat}(2)} \\ K_{p(3)}x_p \\ \vdots \\ K_{p(m)}x_p \end{bmatrix} \; u_5(x_p) = \begin{bmatrix} -u_{\text{sat}(1)} \\ u_{\text{sat}(2)} \\ K_{p(3)}x_p \\ \vdots \\ K_{p(m)}x_p \end{bmatrix} \; ...$$

$$u_{3^m}(x_p) = \begin{bmatrix} K_{p(1)}x_p \\ K_{p(2)}x_p \\ \vdots \\ K_{p(m-1)}x_p \\ u_{\text{sat}(m)} \end{bmatrix}$$

$$x = \begin{cases} x_{S\_1} & \forall x_p \in S_1 \\ x_{S\_2} & \forall x_p \in S_2 \\ \vdots & \\ x_{S\_3^m} & \forall x_p \in S_{3^m} \end{cases} \tag{65}$$

The piece-wise quadratic shape of $\mathcal{E}_{Vp}(P)$ is graphically represented as the union of decentralized ellipsoids and one centered, which can be visualized in Figure 3, in the spaces of $\mathbb{R}^2$ ($n_p = 2$, $m = 1$) and of $\mathbb{R}^{n_p}$. Obviously, it is not possible to completely represent dimensions higher than two in a figure, so the figure with the generic dimension should be interpreted as a pictorial illustration of the 3^m regions and the level sets inside each of them.

Figure 3 – $\mathcal{E}_{Vp}(P)$ in $\mathbb{R}^2$, with $m = 1$ (upper), and in $\mathbb{R}^{n_p}$ (lower).



Source: the author.

After this remark, the next section presents the results of this section applied on the optimization problems.

4.4 Optimization problems

The stability conditions of the previous theorems can be cast as constraints of optimization problems, with the purpose of maximizing the size of the region $\mathcal{E}_V$. The adopted criterion of size (formally called *measure* in the field of mathematics) is the length of the minor axis of the ellipsoid, that is proportional to the maximum eingenvalue of P_0. It corresponds to the last constraints of each problem proposed below, which ensure that the maximal eigenvalue of P_0 is smaller than the objective variable ε that ought to be minimized. Hence, the problem can be thought as the maximization of the smallest of the axes of $\mathcal{E}_V$ subject to the constraints of the theorems developed so far.

The optimization problems with the conditions of theorems 5 and 6 are respectively given as follows:

$$\begin{aligned} &\min_{P_0,P_1,G_j,N,\varepsilon} \varepsilon \\ &\text{subject to:} \\ &(44), (45), (46), (47), \\ &(48), (49), (50), (51) \\ &\text{and } P_0 - \varepsilon I < 0 \end{aligned} \tag{66}$$

$$\begin{aligned} &\min_{P_0,\cdots,P_d,G_j,N,\varepsilon} \varepsilon \\ &\text{subject to:} \\ &(57), (58), (59), (60) \\ &\text{and } P_0 - \varepsilon I < 0 \end{aligned} \tag{67}$$

It was mentioned earlier that either G_j or T needs to be fixed in order that the problems can be solved through SDP (problem (66)) and SOSP (problem (67)) techniques. The variable T is chosen as its dimension is lesser than of G_j if $n > m$, and, moreover, T is a scalar when system (18) is SISO, making the series of executions relatively quick in this case. Observe that fixing these variables would in principle limit the range of possible solutions, making the optimization result in a solution closest to the optimal, but not the optimal itself; However, if a series of optimizations is made over a finite grid containing discrete values of T, the global solution (or an approximation of it) can still be found. To finish the Problem **P1** here, observe that (67) must have as well its polynomial degree d fixed *before* its solution, and so d should be specified *a priori* every time (67) is approached.

Shifting now the attention from Problem **P1** to **P2**, we can define the set of admissible initial conditions $\mathcal{P}$ as the convex hull Co$(\cdot)$ of the points $v_{(i)} \in \mathbb{R}^{n_p}$, $i = 1, \cdots \overline{v}$.

$$\mathcal{P} \triangleq \{x \in \mathbb{R}^{n_p} \; : \; x \in \text{Co}(v_{(i)}), \quad i = 1, \cdots, \overline{v}\} \tag{68}$$

The problem (of estimating the maximum $\overline{\tau}$) is solved iteratively increasing the value of $\overline{\tau}$, while taking into account the inclusion of the convex polyhedron $\mathcal{P}$ in region $\mathcal{E}_{Vp}(P_0)$, which is expressed by (69). A consequence is that the optimization problems are modified to the solution of feasibility problems (70) and (71) over a grid on T and $\overline{\tau}$.

$$\mathcal{P} \subset \mathcal{E}_{Vp}(P_0) \Leftrightarrow \begin{bmatrix} v'_{(i)} & u'_j(v_{(i)}) \end{bmatrix} P_0 \begin{bmatrix} v_{(i)} \\ u_j(v_{(i)}) \end{bmatrix} \leq 1 \qquad i = 1, ..., \overline{v} \tag{69}$$

$$\begin{aligned} &\text{find } P_0, P_1, G_j, N \\ &\text{subject to:} \\ &(44), (45), (46), (47), \\ &(48), (49), (50), (51), \\ &\text{and } (69) \end{aligned} \tag{70}$$

$$\begin{aligned} &\text{find } P_0, \cdots, P_d, G_j, N \\ &\text{subject to:} \\ &(57), (58), (59), (60), \\ &\text{and } (69) \end{aligned} \tag{71}$$

In the next section, these problems are solved by the software compatible with the SDP and SOSP.

4.5 Numerical examples

Some numerical examples are presented here, obtained with the solvers SeDuMi (STURM, 1999) and SOSTools (PRAJNA *et al.*, 2004). Before continuing, recall that the problem (67) needed the polynomial degree d imposed before its solution. Here, in a test similar to one found in BRIAT, 2015, the degree d of the polynomial $P(\tau)$ (of equation (54)) is increased while the impact on ε_{min} is registered.

In principle, with a greater d, more degrees of freedom are contemplated in the solution process, and lesser is the conservatism of the optimal solution, i.e., ε_{min} tends to be lower at the end. It should be noted that, since an increase in d entails a greater computational effort, the trade-off between conservatism and numerical complexity should be mindfully balanced in a general case. Nevertheless, in the particular case of the following examples, the intention is to explore the full potential of the current method in estimating the region of attraction of system $\mathcal{H}$, so the degree d should ideally be as high a value as the limit past which no more reduction in conservatism can be attained.

Consider the following parameters:

$$\begin{aligned} A &= \begin{bmatrix} -0.25 & 1 \\ 1 & -0.25 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \\ K &= \begin{bmatrix} -1.5 & -1 \end{bmatrix} \qquad u_{\text{sat}} = 1; \end{aligned} \tag{72}$$

$$
\begin{aligned}
A &= \begin{bmatrix} -0.5 & 1 \\ 1.5 & -0.25 \end{bmatrix};\ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix};\\
K &= \begin{bmatrix} -1.5 & -2 \end{bmatrix}; \qquad u_{\text{sat}} = 1;
\end{aligned} \tag{73}
$$

Aiming for that limit, the Table 1 shows the values of the objective variable that result from the optimization of (67), always considering $T = 0.1$. In the table, the polynomial degrees in the column of d are respective to the SOS constraints as a whole: no term in a constraint is allowed to have a degree greater than the degree of $P(\tau)$. It means that $Q_1(\tau)$,...,$Q_4(\tau)$ should not have degree greater than $d-2$ (since these terms always multiply τ^2).

Table 1 – Results of problem (67).

d	Param.	δ_k	ε_{min}	d	Param.	δ_k	ε_{min}
2	(72)	$[0.05, 0.1]$	0.7506	2	(72)	$[0.1, 0.15]$	0.5954
4			0.7497	4			0.5935
6			0.7497	6			0.5935
8			0.7497	8			0.5935
2	(73)	$[0.02, 0.04]$	0.6465	2	(73)	$[0.04, 0.06]$	0.5226
4			0.6463	4			0.5222
6			0.6463	6			0.5222
8			0.6463	8			0.5222

When all the resulting ε_{min} are taken into account, we perceive that there is no advantage in choosing any $d > 4$. Even though there are other possibilities of parameters that could be tested, it is assumed from now on as a rule that $d = 4$ is the limit past which the exploration of systems with dimensions $n_p = 2$ and $m = 1$ isn't any better. So, keeping in accordance with this dimension, consider the following parameters:

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ -5 \end{bmatrix};\\
K &= \begin{bmatrix} 2.6 & 1.4 \end{bmatrix} \qquad u_{\text{sat}} = 1;
\end{aligned} \tag{74}
$$

The parameters above are the same that were used by FIACCHINI; GOMES DA SILVA Jr., 2018 to compare their own method to the one of SEURET; GOMES DA SILVA Jr.. The comparison, when reproduced, respectively results in the matrices P_{F} and P_{S}:

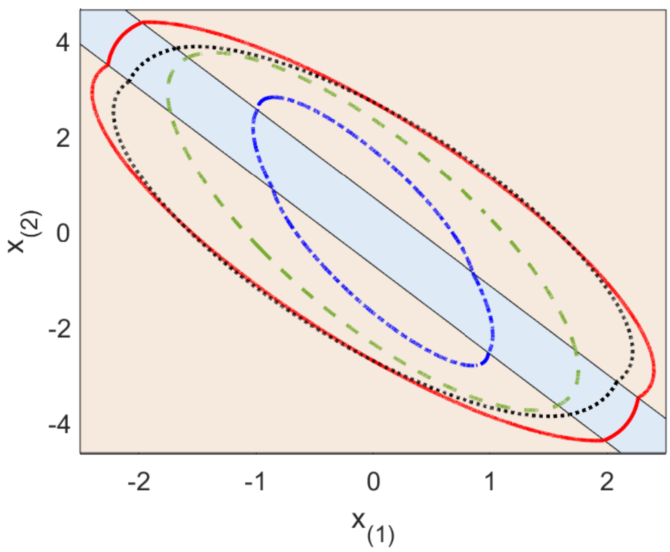
$$
P_{\text{F}} = \begin{bmatrix} 0.5341 & 0.2397 & -0.1242 \\ \star & 0.1811 & -0.0752 \\ \star & \star & 0.0754 \end{bmatrix} \qquad P_{\text{S}} = \begin{bmatrix} 0.8241 & 0.2997 & 0 \\ \star & 0.1802 & 0 \\ \star & \star & 0 \end{bmatrix}
$$

In the next numerical example, the method of analysis developed until now will be added to this comparison. The problems (66) and (67) with a polynomial degree $d = 4$ are respectively solved for $T = 0.2$ and 0.05, which were found to be the optimum values, and for an intersampling time of $[\underline{\tau}, \overline{\tau}] = [0.05, 0.1]$. The objective variable of each solution is respectively minimized to $\varepsilon_{\text{min}} = 4.0784$ and $\varepsilon_{\text{min}} = 0.6927$, and the matrices P_0 are respectively given by (75) and (76). Notice that the big gap between the minimum values ε_{min} of each solution noticeably corresponds to the fact that the estimates $\mathcal{E}_{Vp}(P_0)$ are extremes when they are plotted together with $\mathcal{E}_{Vp}(P_{\text{F}})$ and $\mathcal{E}_{Vp}(P_{\text{S}})$ in Figure 4.

$$P_0 = \begin{bmatrix} 3.4169 & 1.2705 & -0.6855 \\ \star & 0.6611 & -0.3883 \\ \star & \star & 0.3939 \end{bmatrix} \quad \text{(prob. (66))} \tag{75}$$

$$P_0 = \begin{bmatrix} 0.5275 & 0.2552 & -0.1127 \\ \star & 0.1836 & -0.0771 \\ \star & \star & 0.0663 \end{bmatrix} \quad \text{(prob. (67))} \tag{76}$$

Figure 4 – Estimates obtained from problem (66) ($\mathcal{E}_{Vp}(P_0)$, dashed and dotted blue), from problem (67) ($\mathcal{E}_{Vp}(P_0)$, continuous red), from FIACCHINI; GOMES DA SILVA Jr., 2018 ($\mathcal{E}_{Vp}(P_{\text{F}})$, dotted black), and from SEURET; GOMES DA SILVA Jr., 2012 ($\mathcal{E}_{Vp}(P_{\text{S}})$, dashed green). This is the case of parameters (74). Observe that the regions of saturation in this figure have the same colours of the illustrative Figure 3.



Source: the author.

By extremes these estimates are referred to because one encompasses all the other estimates, while another is in turn encompassed by every other. In other words, the problems (66) and (67) respectively provide the most and least conservative estimates, and, all in all, P_0 (from (67)) - P_F - P_S - P_0 (from (66)) is the sequence ordered from the most to the least conservative estimate.

It should be clarified, however, that this relation of conservatism is not the same in all cases. To illustrate this caveat, consider another case:

$$\begin{aligned} A &= \begin{bmatrix} -0.25 & 1 \\ 1 & -0.25 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \\ K &= \begin{bmatrix} -1.5 & -1 \end{bmatrix} \qquad u_{\text{sat}} = 1; \end{aligned} \tag{77}$$

The problems (66) and (67) with a polynomial degree $d = 4$ are optimally solved in this case with $T = 0.06$ and with $T = 0.03$, again with an intersampling time of $[\underline{\tau}, \overline{\tau}] = [0.05, 0.1]$. Their solution results respectively in $\varepsilon_{\text{min}} = 0.6641$ and $\varepsilon_{\text{min}} = 0.495$ and in the matrices P_0 (78) and (79), that follow along with the matrices P_{F} and P_{S} that respectively result from the methods of FIACCHINI; GOMES DA SILVA Jr. and SEURET; GOMES DA SILVA Jr.:

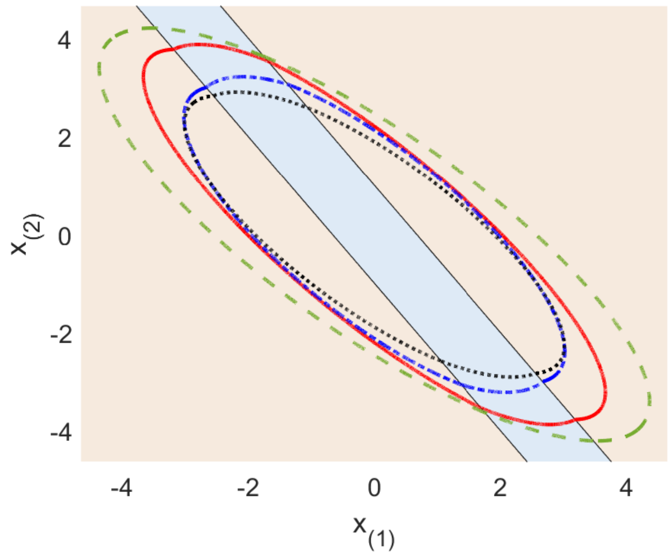
$$P_{\text{F}} = \begin{bmatrix} 0.3303 & 0.2617 & 0.0662 \\ \star & 0.3224 & 0.0568 \\ \star & \star & 0.0623 \end{bmatrix} \qquad P_{\text{S}} = \begin{bmatrix} 0.1528 & 0.1282 & 0 \\ \star & 0.1641 & 0 \\ \star & \star & 0 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} 0.3040 & 0.2242 & 0.0546 \\ \star & 0.2605 & 0.0538 \\ \star & \star & 0.0500 \end{bmatrix} \quad \text{(prob. (66))} \tag{78}$$

$$P_0 = \begin{bmatrix} 0.2825 & 0.2233 & 0.0472 \\ \star & 0.2394 & 0.0468 \\ \star & \star & 0.0400 \end{bmatrix} \quad \text{(prob. (67))} \tag{79}$$

Notice that with those different parameters, the relation of conservatism is entirely different: this time the sequence is, from least to most conservative, P_S - P_0 (from (67)) - P_0 (from (66)) - P_F, as seen in Figure 5. From what have been exposed until now, the only certainty concerning the relation of conservatism associated to the estimates is that the solution derived from the problem (67), that is based on SOSP, will always be less conservative than the solution derived from the problem (66), that is based on SDP, for the reason that the theorem behind (67) can be seen as a generalization of the theorem behind (66).

Figure 5 – Estimates obtained from problem (66) ($\mathcal{E}_{Vp}(P_0)$, dashed and dotted blue), from problem (67) ($\mathcal{E}_{Vp}(P_0)$, continuous red), from FIACCHINI; GOMES DA SILVA Jr., 2018 ($\mathcal{E}_{Vp}(P_{\text{F}})$, dotted black), and from SEURET; GOMES DA SILVA Jr., 2012 ($\mathcal{E}_{Vp}(P_{\text{S}})$, dashed green). This is the case of parameters (77).



Source: the author.

In that last example, the presently developed method unfortunately didn't provide the best estimate. To propose an improvement of this result, consider the optimization criterion given by $P_0 - \varepsilon I < 0$. As stated before, this criterion ensure that the maximal eigenvalue of P_0 is smaller than the objective variable ε, which leads to a general but aimless expansion of the region $\mathcal{E}_{Vp}(P_0) = \{x_p \in \mathbb{R}^{n_p} \; : \; [x_p' \text{ sat}(Kx_p)']P_0[x_p' \text{ sat}(Kx_p)']' \leq 1\}$, since the lengths of its axes are inversely proportional to the eigenvalues of P_0. A criterion that can be used *ad hoc* here as a substitute for $P_0 - \varepsilon I < 0$ is given as follows:

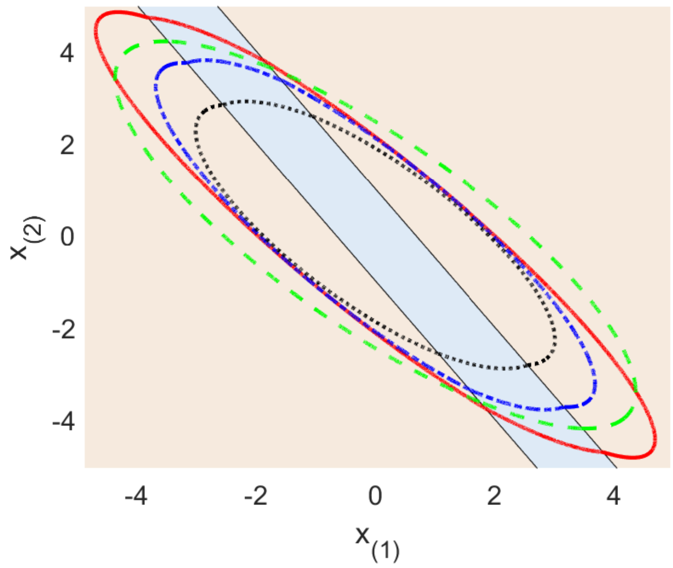
$$l'P_0 l - \varepsilon < 0 \tag{80}$$

This criterion demands a vector $l \in \mathbb{R}^n$ to be fixed *a priori*, which in this case is, after a search for the most suitable, given by $l = [-1 \; 1 \; 1]'$. With this modification and adopting $T = 0.04$ in all problems, the larger $\mathcal{E}_{Vp}(P_0)$ expands and, notably, cover some region of its own, thus breaking the absolute relation of conservatism observed until now. The matrices P_0 that were obtained in this case follow:

$$P_0 = \begin{bmatrix} 0.3095 & 0.2497 & 0.0609 \\ \star & 0.2663 & 0.0560 \\ \star & \star & 0.0523 \end{bmatrix} \quad \text{(prob. (66))} \tag{81}$$

$$P_0 = \begin{bmatrix} 0.2983 & 0.2543 & 0.0551 \\ \star & 0.2570 & 0.0498 \\ \star & \star & 0.0397 \end{bmatrix} \quad \text{(prob. (67))} \tag{82}$$

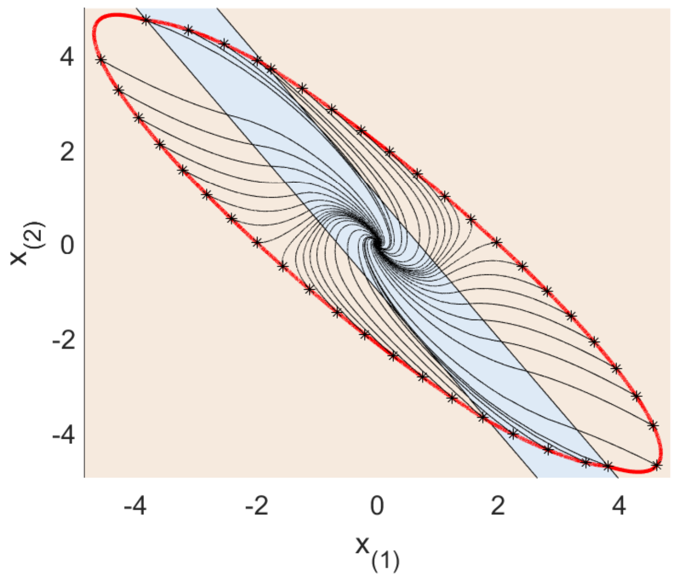
Figure 6 – Estimates obtained from problem (66) ($\mathcal{E}_{Vp}(P_0)$, dashed and dotted blue), from problem (67) ($\mathcal{E}_{Vp}(P_0)$, continuous red), from FIACCHINI; GOMES DA SILVA Jr., 2018 ($\mathcal{E}_{Vp}(P_{\mathrm{F}})$, dashed green), and from SEURET; GOMES DA SILVA Jr., 2012 ($\mathcal{E}_{Vp}(P_{\mathrm{S}})$, dotted black). This is the case of parameters (77) with the modified criterion of size (80).



Source: the author.

The estimate should be thoroughly tested to guarantee that it is a region of stability. To perform this test, a simulation of the system $\mathcal{H}$ is executed for several initial states along the boundary of the larger $\mathcal{E}_{Vp}(P_0)$. Random sampling time sequences are generated before each initial condition is simulated (while respecting $[\underline{\tau}, \overline{\tau}] = [0.05, 0.1]$), i.e. the set Θ is not shared by the individual trajectories, which are shown in Figure 7 with asterisks marking the initial conditions where they start.

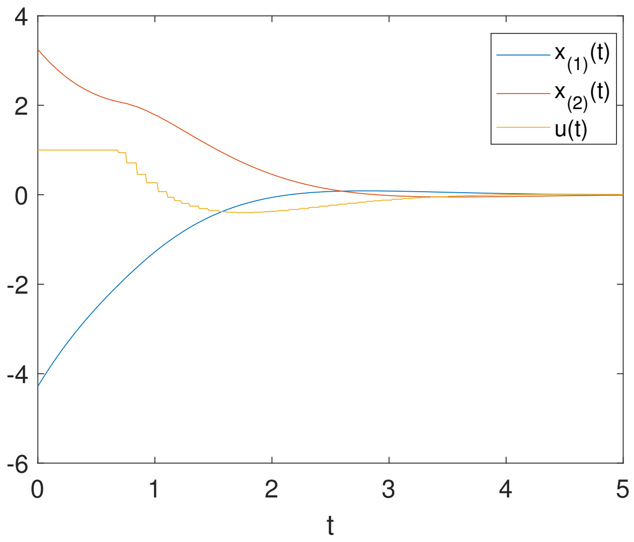
Figure 7 – Trajectories starting at the boundary of $\mathcal{E}_{Vp}(P_0)$ obtained from problem (67) with parameters given by (77), and sampling intervals in the interval $[\underline{\tau}, \overline{\tau}] = [0.05, 0.1]$.



Source: the author.

In the same figure, the regions of saturation are delimited with the colors of Figure 3. It is remarkable how the analysis extends the estimate well beyond the linear region, causing most of the trajectories to be appreciably affected by saturation. Indeed, in Figure 8, one of the trajectories shows that the saturation in control signal $u(t)$ persists for about a quarter of the transient period.

Figure 8 – A simulation with $x(0) = [-4.28\ 3.25]'$.



Source: the author.

4.5.1 Maximization of sampling period respecting a polyhedral inclusion

In the next example, the problem **P2** is tackled by the solution of (70) with the parameteres that follow:

$$\begin{aligned} A &= \begin{bmatrix} -0.25 & 1 \\ 1 & -0.25 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \\ K &= \begin{bmatrix} -1.5 & -1 \end{bmatrix} \qquad u_{\text{sat}} = 1; \end{aligned} \tag{83}$$

The same lower intersampling bound $\underline{\tau} = 0.05$ as before is considered. Two polyhedrons are constructed here, given by $\mathcal{P}_1 = \{x \in \mathbb{R}^{n_p} : \text{Co}(v_{\text{a}(i)}),\ i = 1, \cdots, 6\}$ and $\mathcal{P}_2 = \{x \in \mathbb{R}^{n_p} : \text{Co}(v_{\text{b}(i)}),\ i = 1, \cdots, 6\}$, with points $v_{\text{a}(i)}$ and $v_{\text{b}(i)}$ stated in the Table 2.

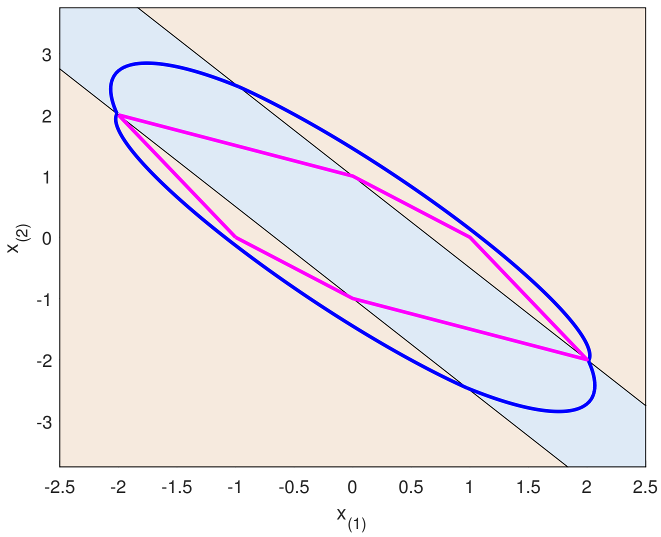
After finding the optimal $T = 0.4$ and $T = 0.1$ respectively for polyhedrons $\mathcal{P}_1$ and $\mathcal{P}_2$, the results are exposed in Table 2. The figures 9 and 10 show that these polyhedrons are encompassed by estimates of the region of attraction, which guarantees that they are regions of stability and finish the problem.

Table 2 – The points of polyhedrons $\mathcal{P}_1$ and $\mathcal{P}_2$, and their resulting $\overline{\tau}$.

Param.	$v_{\mathrm{a}(i)}$	i	$\overline{\tau}$
(83)	[-2, 2]	1	0.2
	[0, 1]	2	
	[1, 0]	3	
	[2, -2]	4	
	[0, -1]	5	
	[-1, 0]	6	

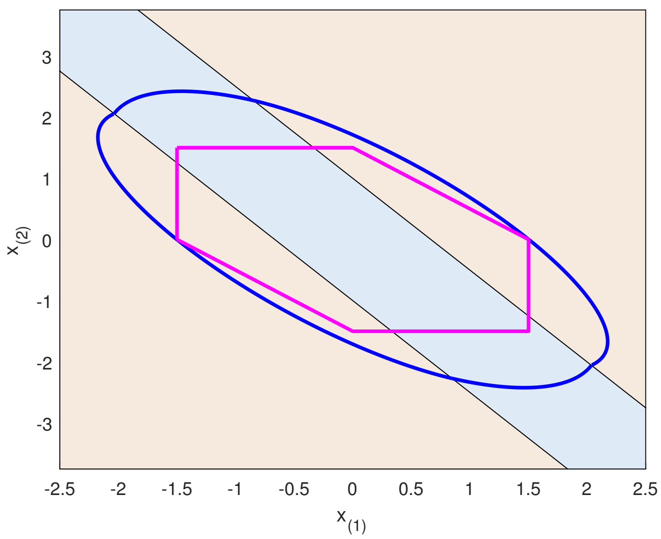
Param.	$v_{\mathrm{b}(i)}$	i	$\overline{\tau}$
(83)	[-1.5, 1.5]	1	0.16
	[0, 1.5]	2	
	[1.5, 0]	3	
	[1.5, -1.5]	4	
	[0, -1.5]	5	
	[-1.5, 0]	6	

Figure 9 – Region of stability encompassing $\mathcal{P}_1$.



Source: the author.

Figure 10 – Region of stability encompassing $\mathcal{P}_2$.



Source: the author.

4.6 Final comments

The contribution of this chapter is an analysis method for the SDS subject to input saturation and aperiodic sampling. The development of the method started with the premise that the quadratic clock-dependent Lyapunov function introduced in Chapter 3 had to be specified to particular candidate functions in order that stability of the system (18) could be assessed in semidefinite or sum-of-squares programmings. The candidate functions with affine dependency on the clock variable allowed the development of stability conditions that could be cast in a semidefinite optimization problem (published in FAGUNDES; GOMES DA SILVA Jr.; JUNGERS, 2019), while the functions with polynomial dependence enabled an optimization in the sum-of-squares programming.

The optimization problems were stated with an objective function such that the process of solving them results in the largest estimate of the region of attraction of the system assessed, or in the maximum intersampling for which a stability region is guaranteed. Numerical examples have confirmed the applicability of the results, with their presence connoting that the method has been successfully implemented in the available software.

5 STABILIZING CONTROLLER SYNTHESIS

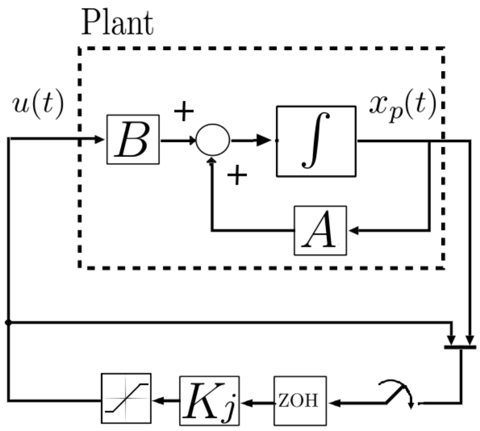
5.1 Introduction

In this chapter, the problem **P3** is tackled. It concerns the synthesis of a controller for system (18), which is graphically represented in Figure 2 as the block K_j. In particular, we consider in this chapter the synthesis of a more generic control law, in which is the value of the control computed in the previous instant is also taken into account, i.e:

$$u^+(t) = \begin{bmatrix} K & K_u \end{bmatrix} \begin{bmatrix} x_p(t) \\ u(t) \end{bmatrix}$$

which corresponds to controllers with a structure $K_j = [K \; K_u]$ that are designed to read their own output when calculating the next input $u(t)$, which can be visualized in the following figure:

Figure 11 – Feedback loop where the controller output is sampled.



Source: adapted from PALMEIRA, 2015.

The preliminary results from Chapter 3 underpin the developments presented here, which are aimed towards sufficient conditions for the constraints of Theorem 3. The main obstacle to this goal is the formulation of design conditions that provide optimization problems as SDP and SOS, in the same manner as the last chapter. In the next section, it is shown how Theorem 4 is modified to circumvent this obstacle.

5.2 Base theorem modification

The problem in using Theorem 4 for the synthesis of a controller comes from the term $N\Gamma$ in constraint (30):

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \begin{bmatrix} -I & \mathbb{A}_j & B_j \end{bmatrix}$$

(recalling that $\mathbb{A}_j = A_j + B_j K_j$)

Specifically, the terms $N_1\mathbb{A}_j$, $N_2\mathbb{A}_j$ and $N_3\mathbb{A}_j$ eliminate the possibility of an optimization in SDP or SOSP when both N and K are decision variables. What follows is a modification of Theorem 4 where these terms are decoupled and the possibility of optimizations is reclaimed, at the cost of some conservatism. The steps for this modification are inspired by SEURET; GOMES DA SILVA Jr., 2012, from which stems the following theorem:

Theorem 7. *If there exist a matrix function $\tilde{P} : [0, \overline{\tau}] \to \mathbb{S}^n$, matrices $\tilde{K}_j$ $\tilde{G}_j \in \mathbb{R}^{m\times n}$, $\tilde{N} \in \mathbb{R}^{n\times n}$, a diagonal matrix $U \in \mathbb{R}^{m\times m}$ and a positive scalar β that satisfy the following inequalities*

$$\tilde{P}(\tau) > 0 \quad \forall\tau \in [0, \overline{\tau}] \tag{84}$$

$$\begin{bmatrix} \dot{\tilde{P}}(\tau) + \beta(A_f\tilde{N}' + \tilde{N}A_f') & \tilde{P}(\tau) - \beta\tilde{N}' + \tilde{N}A_f' \\ \star & -\tilde{N}' - \tilde{N} \end{bmatrix} < 0 \quad \forall\tau \in [0, \overline{\tau}) \tag{85}$$

$$\begin{bmatrix} \tilde{P}(0) - \beta(\tilde{N} + \tilde{N}') & \beta(A_j\tilde{N}' + B_j\tilde{K}_j) - \tilde{N} & \beta B_j U \\ \star & A_j\tilde{N}' + B_j\tilde{K}_j + (A_j\tilde{N}' + B_j\tilde{K}_j)' - \tilde{P}(\tau) & B_j U - \tilde{G}_j' \\ \star & \star & -2U \end{bmatrix} < 0 \quad \forall\tau \in [\underline{\tau}, \overline{\tau}] \tag{86}$$

$$\begin{bmatrix} \tilde{P}(\tau) & (\tilde{K}_{j(i)} - \tilde{G}_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0 \quad \begin{matrix} \forall\tau \in [\underline{\tau}, \overline{\tau}] \\ i = 1, \cdots, m \end{matrix} \tag{87}$$

then, considering $K_j = \tilde{K}_j(\tilde{N}')^{-1}$, $P(0) = (\tilde{N}')^{-1}\tilde{P}(0)\tilde{N}^{-1}$ and a quadratic Lyapunov function $V(\eta(t)) = V(x(t), \tau(t)) = x'(t)P(\tau(t))x(t)$, all initial conditions of system (18) *such that $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n : V(x, 0) = x'P(0)x \leq 1\}$ result in solutions $x(t)$ that converge asymptotically to the origin as $t \to +\infty$.*

Proof. Consider $N = [N_1' \; N_2' \; N_3']'$, with $N_1, N_2 \in \mathbb{R}^{n\times n}$ and $N_3 \in \mathbb{R}^{m\times n}$. Condition (30) from Theorem 4 is written as:

$$\begin{bmatrix} P(0) & 0 & 0 \\ \star & -P(\tau) & -G_j'T \\ \star & \star & -2T \end{bmatrix} + \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \Gamma + \Gamma' \begin{bmatrix} N_1' & N_2' & N_3' \end{bmatrix} < 0 \quad \forall\tau \in [\underline{\tau}, \overline{\tau}] \tag{88}$$

Considering that N_2 is non-singular, equation (88) is pre and post-multiplied by $\text{diag}(N_2^{-1}, N_2^{-1}, T^{-1})$ and its transpose, respectively. The result is the following equivalent condition:

$$\begin{bmatrix} N_2^{-1}(P(0) - \text{He}_1)(N_2')^{-1} & N_2^{-1}N_1\mathbb{A}_j(N_2')^{-1} - N_2^{-1} & N_2^{-1}(N_1B_j - N_3')T^{-1} \\ \star & \text{He}_2 - N_2^{-1}P(\tau)(N_2')^{-1} & B_jT^{-1} + N_2^{-1}(\mathbb{A}_j'N_3'T^{-1} - G_j') \\ \star & \star & \text{He}_3 - 2T^{-1} \end{bmatrix} < 0$$

with

$$\begin{cases} \text{He}_1 = \text{He}\{N_1\} \triangleq N_1 + N_1' \\ \text{He}_2 = \text{He}\{\mathbb{A}_j(N_2')^{-1}\} \triangleq \mathbb{A}_j(N_2')^{-1} + N_2^{-1}\mathbb{A}_j' \\ \text{He}_3 = \text{He}\{N_3B_j\} \triangleq N_3B_j + B_j'N_3' \\ \mathbb{A}_j = A_j + B_jK_j \end{cases}$$

This condition is made stricter by the imposition of new relations $N_1 = \beta N_2$ and $N_3 = 0$. Also, the following variables are introduced: $U = T^{-1}$, $\tilde{N} = N_2^{-1}$, $\tilde{P}(\tau) = \tilde{N}'P(\tau)\tilde{N}$, $\tilde{G}_j = G_j\tilde{N}'$ and $\tilde{K}_j = K_j\tilde{N}'$. The result is the sufficient condition:

$$\begin{bmatrix} \tilde{P}(0) - \beta(\tilde{N} + \tilde{N}') & \beta\tilde{\mathbb{A}}_j - \tilde{N} & \beta B_jU \\ \star & \tilde{\mathbb{A}}_j + \tilde{\mathbb{A}}_j' - \tilde{P}(\tau) & B_jU - \tilde{G}_j' \\ \star & \star & -2U \end{bmatrix} < 0 \quad \forall\tau \in [\underline{\tau},\ \overline{\tau}]$$

Therefore, it is clear that (86) is a sufficient condition for (30).

Next, since we consider[1] the matrix $\tilde{N} = N_2^{-1}$ to be non-singular , it follows that if $\tilde{P}(\tau) > 0$, then $(\tilde{N}')^{-1}\tilde{P}(\tau)\tilde{N}^{-1} = P(\tau) > 0$. The conclusion is that (28) is verified if and only if (84) is verified as well.

Thereafter, the pre and post-multiplication of (31) by $\text{diag}(\tilde{N}, 1)$ and its transpose respectively results in the following equivalent condition:

$$\begin{bmatrix} \tilde{N}'P(\tau)\tilde{N} & \tilde{N}(K_{j(i)} - G_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0 \quad \begin{matrix} \forall\tau \in [\underline{\tau},\ \overline{\tau}] \\ i = 1, \cdots, m \end{matrix}$$

Considering again the identities $\tilde{P}(\tau) = \tilde{N}'P(\tau)\tilde{N}$, $\tilde{G}_j = G_j\tilde{N}'$ and $\tilde{K}_j = K_j\tilde{N}'$, it follows that (31) is verified if and only if (87) is also true.

Considering now that condition (19) in Theorem 3 with $V(x(t), \tau(t)) = x'(t)P(\tau(t))x(t)$ is equivalent to:

$$\begin{gathered} \dot{V}(x(t), \tau(t)) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}' \begin{bmatrix} \dot{P}(\tau(t)) & P(\tau(t)) \\ \star & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} < 0 \\ \forall \begin{bmatrix} A_f & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = 0,\ \forall\tau \in [0,\ \overline{\tau}) \end{gathered} \tag{89}$$

[1]Albeit this consideration has no explicit correspondence in the theorem, it can always be assumed to be true because the LMI (30) implies that $-(\tilde{N} + \tilde{N}') < 0$ and so is verified if and only if $\tilde{N}$ is non-singular.

The Finsler's Lemma (Appendix 7.1, Lemma 4) is now applied with the following instances:

$$\Gamma = [A_f \ -I] \qquad y = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \dot{P}(\tau) & P(\tau) \\ \star & 0 \end{bmatrix}$$

By the Statement 1 and Statement 4 of the lemma, it is concluded that (89) is equivalent to (90):

$$\begin{bmatrix} \dot{P}(\tau(t)) & P(\tau(t)) \\ \star & 0 \end{bmatrix} + N\,[A_f \ -I] + [A_f \ -I]'\,N' < 0 \quad \forall \tau \in [0,\, \overline{\tau}) \tag{90}$$

Considering $N = [N_1' \ N_2']'$, (90) is rewritten as:

$$\begin{bmatrix} \dot{P}(\tau(t)) + N_1 A_f + A_f' N_1' & P(\tau(t)) - N_1 + A_f' N_2' \\ \star & -N_2' - N_2 \end{bmatrix} < 0 \quad \forall \tau \in [0,\, \overline{\tau}) \tag{91}$$

Pre and post-multiplying (91) by diag$(\tilde{N}, \tilde{N})$ and its transpose, assuming the relation $N_1 = \beta N_2$, and then applying the identities $\tilde{N} = N_2^{-1}$ and $\tilde{P}(\tau) = \tilde{N}' P(\tau) \tilde{N}$, it is evidenced that (85) is sufficient for condition (19) of Theorem 3.

The final conclusion is that if the inequalities of Theorem 7 are satisfied, then the inequalities of Theorem 3 are satisfied as well, which finishes the proof. □

This theorem has the same role of Theorem 4 of being the base of the subsequent ones. Thus, the structure of the next sections is a recurrence of what followed Theorem 4, and the same rationale should be expected.

5.3 Affine clock-dependent Lyapunov function

The Lyapunov function candidate involved in the next theorem is the same of Section 4.2 with affine dependence on τ:

$$V(\eta(t)) = x'(t)(P_0 + \tau(t) P_1) x(t) \tag{92}$$

Before we state the theorem, a brief comment on the literature follows. As mentioned previously, in the context of switched systems, this function candidate has been used in (BOYARSKI; SHAKED, 2009), (ALLERHAND; SHAKED, 2011) and (HU *et al.*, 2003). The synthesis problem was also addressed in each of these publications: in ALLERHAND; SHAKED, 2011, a time-varying state-feedback gain was computed for a linear switched system, with parameters that are uncertain and residing within a polytope; in BOYARSKI; SHAKED, 2009, the controller also assumed the form of a time-varying state-feedback gain, computed for a system with uncertain parameters belonging to a polytope; In HU *et al.*, 2003, the sampled-data problem is considered and yet another time-varying controller is synthesized, aiming the minimization of the H_2 norm of a system with norm bounded uncertainties.

A time-varying controller, in our notation system, would be described as follows:

$$u(t) = K(\tau(t))x(t)$$

Note that, in the context of sampled-data systems, this type of controller does not make much sense, as by assumption $\dot{u}(t) = 0$ for $t \in [t_k, t_{k+1})$. Moreover, even if possible, the practical implementation of such control system would be highly questionable.

Returning from this brief review, what follows is the theorem based on the affine candidate:

Theorem 8. *If there exist a matrix function $\tilde{P} : [0,\, \overline{\tau}] \to \mathbb{S}^n$, matrices $\tilde{K}_j\ \tilde{G}_j \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{m\times m}$ and $\tilde{N} \in \mathbb{R}^{n\times n}$, a diagonal matrix $U \in \mathbb{R}^{m\times m}$ and a positive scalar β that satisfy the following inequalities*

$$\tilde{P}_0 > 0 \tag{93}$$

$$\tilde{P}_0 + \overline{\tau}\tilde{P}_1 > 0 \tag{94}$$

$$\begin{bmatrix} \tilde{P}_1 + \beta(A_f\tilde{N}' + \tilde{N}A_f') & \tilde{P}_0 - \beta\tilde{N}' + \tilde{N}A_f' \\ \star & -\tilde{N}' - \tilde{N} \end{bmatrix} < 0 \tag{95}$$

$$\begin{bmatrix} \tilde{P}_1 + \beta(A_f\tilde{N}' + \tilde{N}A_f') & (\tilde{P}_0 + \overline{\tau}\tilde{P}_1) - \beta\tilde{N}' + \tilde{N}A_f' \\ \star & -\tilde{N}' - \tilde{N} \end{bmatrix} < 0 \tag{96}$$

$$\begin{bmatrix} \tilde{P}_0 - \beta(\tilde{N} + \tilde{N}') & \beta\tilde{\mathbb{A}}_j - \tilde{N} & \beta B_j U \\ \star & \tilde{\mathbb{A}}_j + \tilde{\mathbb{A}}_j' - \tilde{P}_0 - \underline{\tau}\tilde{P}_1 & B_j U - \tilde{G}_j' \\ \star & \star & -2U \end{bmatrix} < 0 \tag{97}$$

$$\begin{bmatrix} \tilde{P}_0 - \beta(\tilde{N} + \tilde{N}') & \beta\tilde{\mathbb{A}}_j - \tilde{N} & \beta B_j U \\ \star & \tilde{\mathbb{A}}_j + \tilde{\mathbb{A}}_j' - \tilde{P}_0 - \overline{\tau}\tilde{P}_1 & B_j U - \tilde{G}_j' \\ \star & \star & -2U \end{bmatrix} < 0 \tag{98}$$

$$\begin{bmatrix} \tilde{P}_0 + \underline{\tau}\tilde{P}_1 & (\tilde{K}_{j(i)} - \tilde{G}_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0 \tag{99}$$

$$\begin{bmatrix} \tilde{P}_0 + \overline{\tau}\tilde{P}_1 & (\tilde{K}_{j(i)} - \tilde{G}_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} \geq 0 \qquad i = 1, \cdots, m \tag{100}$$

$$\tilde{\mathbb{A}}_j = A_j\tilde{N}' + B_j\tilde{K}_j$$

then, considering $K_j = \tilde{K}_j(\tilde{N}')^{-1}$ and $P_0 = (\tilde{N}')^{-1}\tilde{P}_0\tilde{N}^{-1}$ and $P_1 = (\tilde{N}')^{-1}\tilde{P}_1\tilde{N}^{-1}$ and the quadratic Lyapunov function given by (92)*, all initial conditions of system* (18) *such that* $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n \ : \ V(x,0) = x'P_0x \leq 1\}$ *result in solutions* $x(t)$ *that converge asymptotically to the origin as* $t \to +\infty$.

Proof. If the same steps in the proof of Theorem 5 are repeated, it is evidenced that the inequalities of this theorem are sufficient for the verification of the ones in Theorem 7:

$$\begin{aligned} &(93) \text{ and } (94) \Rightarrow (84) \quad ; \quad (95) \text{ and } (96) \Rightarrow (85) \\ &(97) \text{ and } (98) \Rightarrow (86) \quad ; \quad (99) \text{ and } (100) \Rightarrow (87) \end{aligned} \tag{101}$$

□

As in the previous chapter, the LF candidate of (92) shall be extended to a polynomial dependency on τ in the next section.

5.4 Clock-dependent Lyapunov function with polynomial dependence on time

As in section 4.3, we consider here a Lyapunov function $V(x(t),\tau(t)) = x(t)P(\tau(t))x(t)$ with $P(\tau)$ having a polynomial dependence on τ, i.e.

$$P(\tau) = \sum_{i=0}^{d} P_i\tau^i = P_0 + \tau P_1 + \cdots + \tau^d P_d \tag{102}$$

What follows is the theorem for this case.

Theorem 9. *If there exist a matrix function* $\tilde{P} : [0, \overline{\tau}] \to \mathbb{S}^n$*, matrices* $\tilde{K}_j \ \tilde{G}_j \in \mathbb{R}^{m\times n}$*,* $\tilde{N} \in \mathbb{R}^{n\times n}$*, a diagonal matrix* $U \in \mathbb{R}^{m\times m}$ *and positive scalars* β *and* γ *that satisfy the following conditions*

$$Q_1(\tau), Q_2(\tau), Q_3(\tau), Q_4(\tau) \text{ are SOS} \tag{103}$$

$$\tilde{P}(\tau) - \gamma I - Q_1(\tau)\tau(\overline{\tau}-\tau) \text{ is SOS} \tag{104}$$

$$-\begin{bmatrix} \dot{\tilde{P}}(\tau) + \beta(A_f\tilde{N}' + \tilde{N}A_f') & \tilde{P}(\tau) - \beta\tilde{N}' + \tilde{N}A_f' \\ \star & -\tilde{N}' - \tilde{N} \end{bmatrix} - \gamma I - Q_2(\tau)\tau(\overline{\tau}-\tau) \text{ is SOS} \tag{105}$$

$$-\begin{bmatrix} \tilde{P}(0) - \beta(\tilde{N} + \tilde{N}') & \beta\tilde{\mathbb{A}}_j - \tilde{N} & \beta B_j U \\ \star & \tilde{\mathbb{A}}_j + \tilde{\mathbb{A}}_j' - \tilde{P}(\tau) & B_j U - \tilde{G}_j' \\ \star & \star & -2U \end{bmatrix} - \gamma I - Q_3(\tau)\tau(\overline{\tau}-\tau) \text{ is SOS} \tag{106}$$

$$\begin{bmatrix} \tilde{P}(\tau) & (\tilde{K}_{j(i)} - \tilde{G}_{j(i)})' \\ \star & u^2_{\text{sat}(i)} \end{bmatrix} - Q_4(\tau)\tau(\overline{\tau}-\tau) \quad \text{is SOS} \quad i = 1, \cdots, m \tag{107}$$

$$\tilde{\mathbb{A}}_j = A_j\tilde{N}' + B_j\tilde{K}_j$$

then, considering $K_j = \tilde{K}_j(\tilde{N}')^{-1}$ *and* $P(\tau) = (\tilde{N}')^{-1}\tilde{P}(\tau)\tilde{N}^{-1}$ *and a quadratic Lyapunov function* $V(\eta(t)) = V(x(t), \tau(t)) = x'(t)P(\tau(t))x(t)$ *with* $P : \mathbb{R} \to \mathbb{S}^n$ *given by* (102)*, all initial conditions of system* (18) *such that* $x(0) \in \mathcal{E}_V = \{x \in \mathbb{R}^n : V(x, 0) = x'P(0)x \leq 1\}$ *result in solutions* $x(t)$ *that converge asymptotically to the origin as* $t \to +\infty$*.*

Proof. If the same steps in the proof of Theorem 6 are repeated, it follows that the constraints of this theorem are sufficient for the verification of the ones in Theorem 7:

$$\begin{aligned} (104) \Rightarrow (84) \quad &; \quad (105) \Rightarrow (85) \\ (106) \Rightarrow (86) \quad &; \quad (107) \Rightarrow (87) \end{aligned} \tag{108}$$

□

5.5 Optimization problems

Considering the conditions expressed in Theorem 4 and 5, the criterion of size for $\mathcal{E}_V$ in Section 4.4 is not valid here because P_0 is no longer a decision variable. To set an upper bound on the eigenvalues of P_0, the following lemma is adapted from SEURET; GOMES DA SILVA Jr., 2012:

Lemma 3. *Let* β *be a positive scalar,* P_0 *be a positive definite matrix and* $\tilde{N}$ *be a nonsingular matrix, and define* $\tilde{P}_0 = \tilde{N}'P_0\tilde{N}$*. If the following inequality is verified*

$$\begin{bmatrix} \beta^{-2}\varepsilon I & I \\ I & \beta(\tilde{N} + \tilde{N}') - \tilde{P}_0 \end{bmatrix} > 0 \tag{109}$$

then $P_0 < \varepsilon I$*.*

Proof. If $P_0 > 0$ it follows that $(\beta P_0^{-1} - \tilde{N})'P_0(\beta P^{-1} - \tilde{N}) \geq 0$, which implies that $\beta^2 P_0^{-1} \geq (\beta(\tilde{N} + \tilde{N}') - \tilde{P}_0)$, or equivalently, $\beta^{-2}P_0 \leq (\beta(\tilde{N} + \tilde{N}') - \tilde{P}_0)^{-1}$. Hence, from Schur's complement (Appendix 7.3), if (109) is verified, it follows that $\beta^{-2}\varepsilon I > (\beta(\tilde{N} + \tilde{N}') - \tilde{P}_0)^{-1}$, which implies that $\beta^{-2}P_0 < \beta^{-2}\varepsilon I$, and $P_0 < \varepsilon I$. □

It should be remarked that the size criterion given by Lemma 3 is not found in the literature, as it is an extension of the lemma used by SEURET; GOMES DA SILVA Jr., 2012 that was utterly necessary for the feasibility of the problems that are about to be presented.

As in Section 4.4, the problems are still solved iteratively over a grid with values of a parameter that cannot be set as decision variable. More specifically, each optimization problem of this section must be solved with the value from a grid with discrete values of β. The optimization problems corresponding to the conditions of theorems 8 and 9 are, respectively:

$$\begin{aligned} &\min_{\tilde{P}_0,\tilde{P}_1,\tilde{G}_j,\tilde{K}_j,U,\tilde{N},\varepsilon} \quad \varepsilon \\ &\text{subject to:} \\ &(93), (94), (95), (96), \\ &(97), (98), (99), (100) \\ &\text{and } (109) \end{aligned} \tag{110}$$

and

$$\begin{aligned} &\min_{\tilde{P}_0,\cdots,\tilde{P}_d,\tilde{G}_j,\tilde{K}_j,U,\tilde{N},\varepsilon} \quad \varepsilon \\ &\text{subject to:} \\ &(104), (105), (106), (107) \\ &\text{and } (109) \end{aligned} \tag{111}$$

Remark 1. *When the intention is to synthetize a controller for the feedback loop of Figure 2, the transformation $\tilde{K}_j = K_j\tilde{N}'$ must yield a controller of the form $K_j = [K\ 0]$. To this end, it suffices to impose the following structures for $\tilde{N}$ and $\tilde{K}$ in the conditions of theorems 8 and 9:*

$$\tilde{N} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ 0 & \tilde{N}_{22} \end{bmatrix} \qquad \tilde{K}_j = \begin{bmatrix} \tilde{K}_p & 0 \end{bmatrix} \tag{112}$$

5.6 Numerical examples

Consider the same system in the first example in Section 4.5, given by

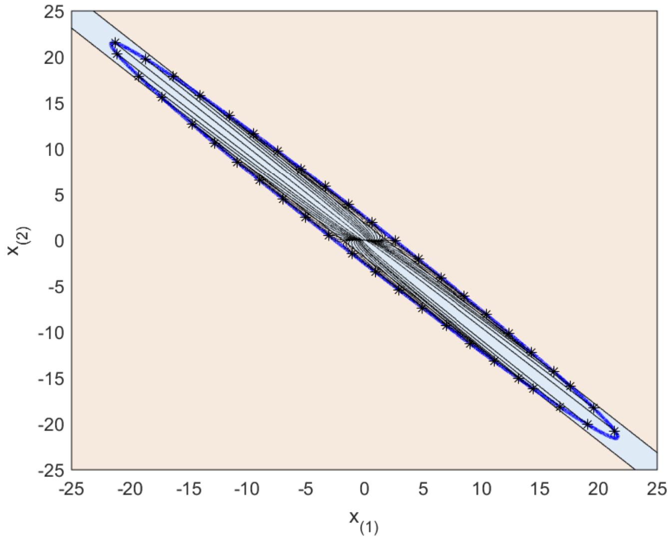
$$A = \begin{bmatrix} -0.25 & 1 \\ 1 & -0.25 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \\ u_{sat} = 1; \tag{113}$$

5.6.1 Structured controller gain

A controller depending only on the plant states is sought, so $\tilde{N}$ and $\tilde{K}_j$ are structured by the equations in (112). The problem (110) results, for the parameters given by (113) and the intersampling range of $\delta_k \in [0.05, 1]$ and for $\beta = 10$, in a solution where the objective variable is minimized to $\varepsilon_{\text{min}} = 0.6504$, and where the optimum gain is found to be $K = [-0.536, -0.536]$. Also obtained from this solution, the matrix $P_0 = (\tilde{N}')^{-1}\tilde{P}_0\tilde{N}^{-1}$ that is given by equation (114) results in the estimated region of attraction $\mathcal{E}_{Vp}(P_0)$ plotted in Figure 12. From the simulations started at the boundary of this region, the one starting on $x_0 = [-6, 8]'$ is plotted in Figure 13.

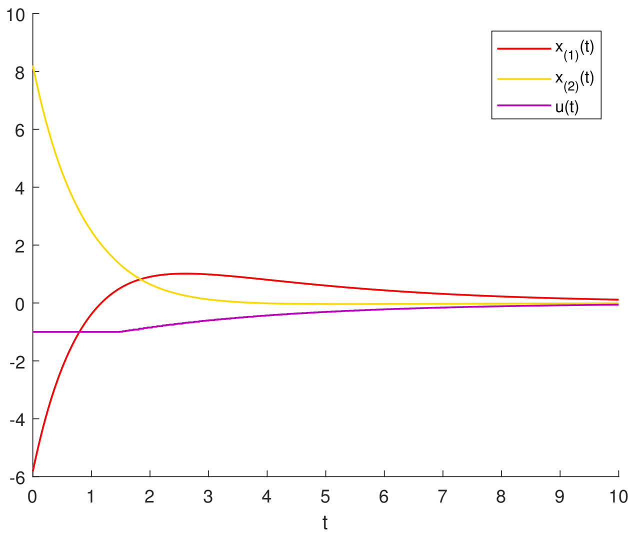
$$P_0 = \begin{bmatrix} 0.2025 & 0.2016 & 0.1007 \\ \star & 0.2030 & 0.1005 \\ \star & \star & 0.2032 \end{bmatrix} \tag{114}$$

Figure 12 – The region $\mathcal{E}_{Vp}(P_0)$ obtained from solving problem (110) and respecting a structured controller, with trajectories starting from its boundary.



Source: the author

Figure 13 – A simulation with $x_0 = [-6, 8]'$.



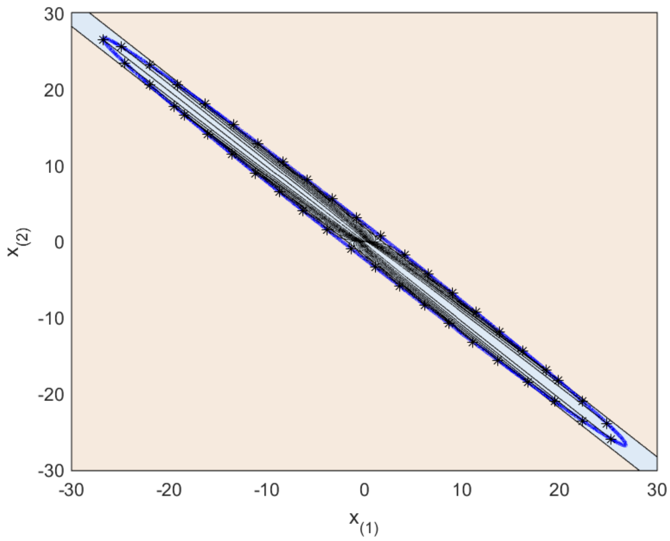
Source: the author

5.6.2 Full matrix controller gain

When the controller to be found is not limited by any particular structure, the problem (110) results, for the same parameters as in the structured case, in a solution where the objective variable is minimized to $\varepsilon_{\text{min}} = 0.5508$ (an expectedly lower value than in the case of structured gain), and where the optimum gain is found to be $K = [-0.556, -0.554, -0.006]$. Again obtained from the solution, the matrix P_0 that is given by equation (115) results in the estimated region of attraction $\mathcal{E}_{Vp}(P_0)$ plotted in Figure 14.

$$P_0 = \begin{bmatrix} 0.2424 & 0.2419 & 0.1016 \\ \star & 0.2427 & 0.1014 \\ \star & \star & 0.1870 \end{bmatrix} \tag{115}$$

Figure 14 – The region $\mathcal{E}_{Vp}(P_0)$ obtained from solving problem (110) and allowing a full matrix controller gain, with trajectories starting from its boundary.



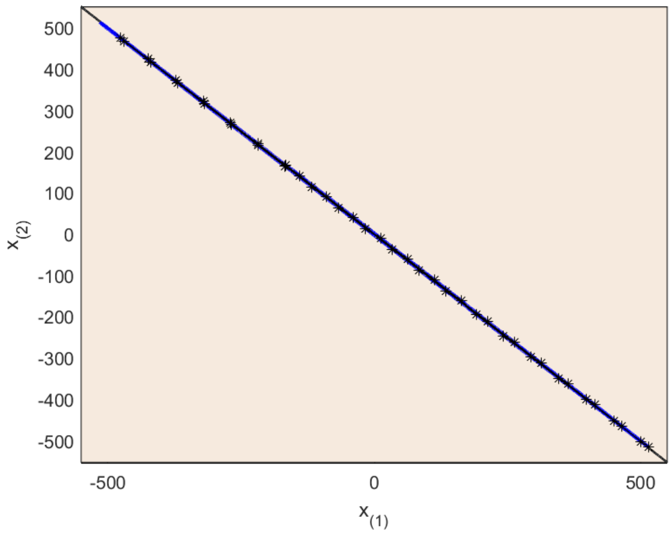
Source: the author

5.6.3 Analysis of the synthesized feed-back system

To further extend the estimation of the region of attraction, the hybrid system $\mathcal{H}$ with parameters given by (113) and the controller $K = [-0.556, -0.554, -0.006]$ obtained in the last example are given as the input of problem (66). Considering these inputs and $T = 0.1$, the objective variable is minimized to an even lower value of $\varepsilon_{\min} = 0.4436$. The estimated region of attraction is plotted in the Figure 15, from the matrix P_0 given by (116). The tendency of the estimate of extending over the linear region suggests that the controller may be designed to guarantee the convergence of all trajectories that start in it.

$$P_0 = \begin{bmatrix} 0.2076 & 0.2077 & 0.0685 \\ \star & 0.2078 & 0.0685 \\ \star & \star & 0.1103 \end{bmatrix} \tag{116}$$

Figure 15 – Region $\mathcal{E}_{Vp}(P_0)$ obtained from solving problem (66) with the parameters (113) controller $K = [-0.556, -0.554, -0.006]$, with trajectories starting from its boundary.



Source: the author

5.7 Final comments

This chapter presented the last contributions of this dissertation. After a modification in the results of the previous chapter, new conditions of stability that enable the synthesis of a stabilizing through the SDP and SOSP have been proposed in this chapter.

The synthesis method presented here, in the same way of the analysis, can be seen as the process of solving the optimization problems of each programming (SDP or SOS). The controller in the closed-loop system considered in Chapter 4 was introduced in this chapter with the possibility of having additional inputs and reading its own output i.e. of using the information of the control input that was being applied before the sampling. It should be intuitive by now, since arguments of this nature have already been made several times, that the controller with additional inputs should facilitate the process of solving the problems (in the sense that the set of feasible solutions is broader in this case). Conversely, in the case of a controller that exclusively reads the state of the plant, restrictions on the structure of some of the decision variables need to be imposed, which should naturally lead to an increased conservatism.

6 CONCLUSIONS

This dissertation presented a new method for the stability analysis and synthesis of linear sampled-data systems subject to input saturation and aperiodic sampling. This method is based on a hybrid system framework that models the dynamical behaviour of the SDS, and on a combined use of the clock dependent Lyapunov function with the generalized sector condition. Although each chapter of this dissertation ends with a final commentary, the main conclusions from their results are recapitulated in what follows, and then future works are proposed.

In Chapter 3, a representation of the SDS was given in the framework of hybrid systems. The chapter detailed the hypotheses related to the quadratic form of the Lyapunov function that depends on the clock variable of the hybrid system representation, and the relation between the estimate of the region of attraction of the representation and the generalized sector condition. From the hypotheses were developed the generic stability conditions of Theorem 4 that precede, as preliminary results, the analysis method.

In Chapter 4, additional hypotheses regarding the Lyapunov candidate function enabled the preliminary conditions of Theorem 4 to be cast in semidefinite and sum-of-squares optimization problems that constitute the analysis method. Specifically, these hypotheses define the dependence of the Lyapunov candidate on the clock variable as affine or polynomial. The method was tested in numerical examples to provide the largest estimate of the region of attraction, and also to provide the maximum intersampling time for which a polyhedral region of stability is guaranteed. Also, the method was compared to the ones of FIACCHINI; GOMES DA SILVA Jr., 2018 and SEURET; GOMES DA SILVA Jr., 2012, which provided insights about how conservatism with relation to other methods can change when different parameters are considered.

In Chapter 5, the Theorem 4 was reformulated into Theorem 7 by the application of the Finsler's lemma with an appropriate structure of multipliers. The modified theorem allows the synthesis of a control structure that reads both the samples of the plant state and of the control output. Note that this feature is not exposed until Chapter 5 for the sake of maintaining the presentation flow, and there is no difficulty in applying the analysis method of the previous chapter in systems with a controller presenting this structure.

As future works, some themes that could be explored are the following:

1) The synthesis of dynamic output feedback controllers. This extension could be explored considering a more realistic hypothesis that the controller is described by discrete dynamical system, and that the information available for sampling is reduced to what is in the output of the plant.

2) A extension to systems with uncertain parameters. This extension would be more or less difficult considering polytopic or norm bounded uncertainties, respectively. The extension to polytopic uncertainties is actually straightforward from a theoretical viewpoint, as it requires only a repetition of the conditions in the vertices of a polytope. From a practical viewpoint, however, one concern to be regarded is the increased difficulty of finding feasible solutions.

3) A sampled-data system with delay in the control signal. This extension would represent more accurately what a networked control system is, since the delay induced by the communication protocol, along with the uncertain sampling intervals already considered, is a major aspect of this topology.

7 APPENDIX - BASIC CONCEPTS AND INSTRUMENTAL LEMMAS

7.1 Finsler's Lemma

Lemma 4. (FINSLER, 1937) *For* $y \in \mathbb{R}^{2n+m}$, $\Lambda = \Lambda' \in \mathbb{R}^{2n+m \times 2n+m}$ *and* $\Gamma \in \mathbb{R}^{n \times 2n+m}$ *with rank*$(\Gamma) < n$*. The following statements are equivalent:*

1. $y'\Lambda y < 0 \quad \forall y \neq 0 \ : \ \Gamma y = 0$
2. $\tilde{\Gamma}'\Lambda\tilde{\Gamma} < 0 \quad \tilde{\Gamma} \neq 0 \ : \ \Gamma\tilde{\Gamma} = 0$
3. $\exists \rho \in \mathbb{R} \quad : \ \Lambda + \rho\Gamma'\Gamma < 0$
4. $\exists N \in \mathbb{R}^{2n+m \times n} \ : \ \Lambda + N\Gamma + \Gamma' N' < 0$

7.2 Polynomials, semidefinite programming and sum of squares programming

The verification of nonnegativity in a function plays a key role in the present dissertation. Mathematically expressed with some scalar $b > 0$, the condition for this property is stated as:

$$f(x) \geq b, \quad \forall x \in \mathbb{R}^z \tag{117}$$

Beyond the scope of this study, many areas of mathematics encounter the same problem. In every case, to obtain equivalent conditions or a procedure for checking inequality (117), it is necessary to limit the structure of the possible functions f, while at the same time making the problem general enough to guarantee the applicability of the results. A good compromise is achieved by considering the case of polynomial functions (PARRILO, 2000).

A (multivariate) polynomial $f^d : \mathbb{R}^z \to \mathbb{R}$ of degree d is a finite linear combination of monomials:

$$f^d(x) = \sum_{0 \le i_1 + i_2 + ... + i_z \le d} \omega_{\mathbb{I}} x^{\mathbb{I}} \tag{118}$$

where $\mathbb{I} = \{i_1, i_2, ..., i_z\}$ is a *z-tuple*, $\omega_{\mathbb{I}} \in \mathbb{R}$, and $x^{\mathbb{I}} = x_{(1)}^{i_1} x_{(2)}^{i_2} ... x_{(z)}^{i_z} \in \mathbb{R}$. In particular, a polynomial is called a *form* $\hat{f}^d(x)$ when the combination is $i_1 + i_2 + ... + i_z = d$. In the case of a form of degree 2, or quadratic form, the use of *semidefinite programming* (SDP) provides constraints which are convex on the decision variable, and hence result in easy-to-solve optimization problems. More concretely, a general semidefinite programming problem can be defined as any optimization problem formulated as

$$\begin{aligned} & \min_{x \in \mathbb{R}^z} \hat{f}_0^2(x) \\ & \text{subject to:} \\ & \hat{f}_k^2(x) \ge b_k \qquad k = 1, ..., \overline{k} \end{aligned} \tag{119}$$

Considering that $\hat{f}^2(x)$ may be rewritten to

$$\hat{f}^2(x) = \begin{bmatrix} x_{(1)} & x_{(2)} & \cdots & x_{(z)} \end{bmatrix} \Omega \begin{bmatrix} x_{(1)} \\ x_{(2)} \\ \vdots \\ x_{(z)} \end{bmatrix}$$

where $\Omega \in \mathbb{S}^z$ is given by

$$\Omega = \begin{bmatrix} \omega_{2,0,...,0} & \frac{\omega_{1,1,...,0}}{2} & \cdots & \frac{\omega_{1,0,...,1}}{2} \\ \frac{\omega_{1,1,...,0}}{2} & \omega_{0,2,...,0} & \cdots & \frac{\omega_{0,1,...,1}}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\omega_{1,0,...,1}}{2} & \frac{\omega_{0,1,...,1}}{2} & \cdots & \omega_{0,0,...,2} \end{bmatrix}$$

Ω can be expressed as the linear combination of matrices that form its basis, given by $\{\Omega_i \in \mathbb{S}^z : i = 1, ..., g\}$. This expression of Ω is written as $\tilde{\Omega} : \mathbb{R}^g \to \mathbb{S}^z$:

$$\Omega = \sum_{i=1}^{g} y_{(i)} \Omega_i \triangleq \tilde{\Omega}(y)$$

with $g \le z(z+1)/2$, and $y \in \mathbb{R}^g$.

The problem (119) may be reformulated with the substitution of $\hat{f}^2$ by $\tilde{\Omega}$:

$$\begin{aligned} & \min_{x \in \mathbb{R}^g} x' \tilde{\Omega}_0(y) x \\ & \text{subject to:} \\ & x' \tilde{\Omega}_k(y) x \ge b_k \qquad k = 1, ..., \overline{k} \end{aligned} \tag{120}$$

The equivalency between (119) and (120) is maintained for any chosen x. The possibility of x having *any* value brings the concept of the definiteness of a matrix, which is expressed as

$$x'Ax \geq x'Bx \quad \forall x \quad \Leftrightarrow \quad A \succeq B$$

or, to simplify notation, it is stated that $A \geq B$, despite nor A or B being scalars.

Changing the decision variable from x to y, and considering $c = [x'\Omega_1 x, ..., x'\Omega_g x]$ and $x'X_k x = b_k$, the optimization problem (120) assumes the dual-SDP formulation

$$\begin{aligned} &\min_{y \in \mathbb{R}^g} cy \\ &\text{subject to:} \\ &\tilde{\Omega}_k(y) \geq X_k \qquad k = 1, ..., \overline{k} \end{aligned}$$

If we rewrite the last inequality to diag$(\tilde{\Omega}_1(y), ..., \tilde{\Omega}_{\overline{k}}(y)) \geq$ diag$(X_1, ..., X_{\overline{k}})$, we end with a more familiar description of the dual semidefinite programming, constrained by a *linear matrix inequality*:

$$\begin{aligned} &\min_{y \in \mathbb{R}^g} cy \\ &\text{subject to:} \\ &\sum_{i=1}^{g} y_{(i)} \Omega_i \geq X \end{aligned} \tag{121}$$

where

$$\sum_{i=1}^{g} y_{(i)} \Omega_i = \text{diag}(\tilde{\Omega}_1(y), ..., \tilde{\Omega}_{\overline{k}}(y)), \qquad X = \text{diag}(X_1, ..., X_{\overline{k}}) \tag{122}$$

Semidefinite programming (SDP) is an important tool for the analysis of stability in control theory. The most prominent examples of this formulation are problems with constraints in terms of a Lyapunov function. There are some cases, however, where the constraints presented by a problem cannot be cast in terms of LMIs. These cases happen when such constraints do not belong to the subspace spanned by the basis $\{\Omega_i : i = 1, ..., g\}$, i.e., the constraints cannot be expressed as $y_{(1)}\Omega_1 + ... + y_{(g)}\Omega_g \geq X$.

With the intent of covering these cases, consider that a simple condition for a polynomial to be positive and satisfy (117) is that its degree must be even. This condition is met when there exists a *sum of squares decomposition* for this polynomial, that is, when a real-valued function $H_{\text{SOS}} : \mathbb{R}^z \to \mathbb{S}^h$ can be written as

$$H_{\text{SOS}}(x) = H'(x)H(x) \tag{123}$$

When a polynomial can be decomposed into a sum of squares, it is abbreviated that the polynomial *is* a sum of squares or, abbreviating more, that it is SOS.

A curious fact of this decomposition is that it is just sufficient for the positive-definiteness of the polynomial, in contrast to the decomposition of a positive definite matrix (Cholesky decomposition). More than a century ago, the mathematician David Hilbert studied the converse problem and published it in his famous list of the twenty-three unsolved problems as the following question:

- Given a multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions?

Hilbert himself noted that not every positive semidefinite polynomial can be decomposed to a sum of squares. A simple counterexample is the Motzkin form, given by (124). Nonetheless, Hilbert showed that every homogeneous polynomial in n variables and degree d can be represented as sum of squares of other polynomials if and only if either (a) $z = 2$ or (b) $d = 2$ or (c) $z = 3$ and $d = 4$

$$\hat{f}^{6}_{\text{Motzkin}}(x) = x_{(1)}^4 x_{(2)}^2 + x_{(1)}^2 x_{(2)}^4 + x_{(3)}^6 - 3x_{(1)}^2 x_{(2)}^2 x_{(3)}^2 \tag{124}$$

In more recent times, the sum of squares programming has been applied in the search of polynomial Lyapunov functions for dynamical systems described by polynomial vector fields (see, for instance, (TAN; PACKARD, 2004) and (CHAKRABORTY *et al.*, 2011)). In this sense, VALMORBIDA; TARBOURIECH; GARCIA, 2013 used the SOS formulation for the design of polynomial control laws subject to saturation, and VALMORBIDA; ANDERSON, 2017 have derived conditions to guarantee that trajectories started in a positively invariant set converge to the equilibrium point of this type of system, with a method based on SOS programming.

The basic idea of the sum of squares programming (SOSP) is to express a given polynomial constraint as Equation (123). To ellaborate this idea, we express the polynomial from (118) as the product of a coefficients and a monomials vector, with a notation inspired in VALMORBIDA; TARBOURIECH; GARCIA, 2013:

$$f^d(x) = \sum_{0 \le i_1 + i_2 + \ldots + i_z \le d} \omega_{\mathbb{I}} x^{\mathbb{I}} = F\xi(x) \tag{125}$$

In this case, $f^d : \mathbb{R}^z \to \mathbb{R}$ is a scalar-valued polynomial. Considering a matrix polynomial $H : \mathbb{R}^z \to \mathbb{R}^{s \times h}$, in this case the expression is given by (127), where $F \in \mathbb{R}^{s \times \sigma(s,d)}$ is the matrix of the coefficients and $\xi : \mathbb{R}^z \to \mathbb{R}^{\sigma(s,d) \times h}$ is a matrix polynomial with all the σ combinations of monomials that compose it, and $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by:

$$\sigma(s,d) = \frac{(s+d)!}{s!d!} - 1 \tag{126}$$

$$H(x) = F\xi(x) \tag{127}$$

Going back to the case uncovered by LMIs, a possible relaxation of the LMI constraint in (121) is given by (128). In the equation, $\tilde{Q} : \mathbb{R}^g \to \mathbb{S}^z$ is the span of $F'F$, and is given by (129). Since the only property always verified in $F'F$ is that it is positive definite, the exigency of this type of constraint could be described as:

- Given a properly chosen vector of monomials $\xi(x)$, find y such that $\tilde{Q}(y)$ is a positive definite matrix

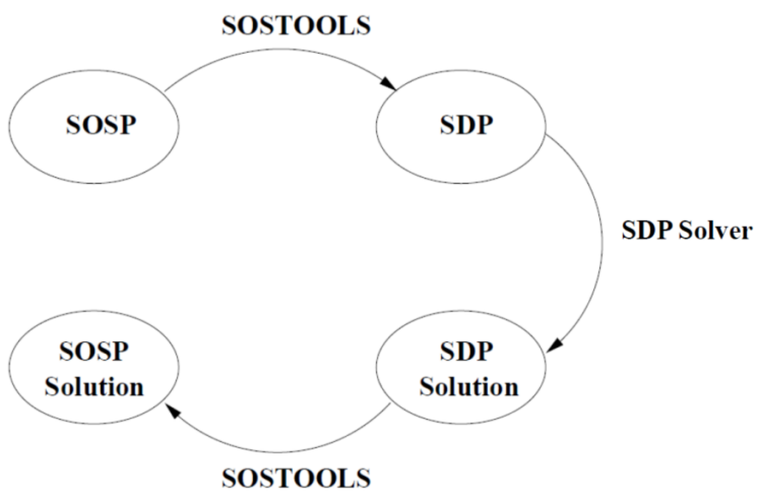
The process of meeting this exigency and expressing an SOS polynomial as $\xi(x)\tilde{Q}(y)\xi(x)$ has also been referred to as the Gram matrix method (PRAJNA *et al.*, 2004, CHOI; LAM; REZNICK, 1995, POWERS; WöRMANN, 1998).

Since the problem of finding the positive definite matrix has received a great deal of attention and is now efficiently dealt with by SDP solvers, the part of finding a "properly chosen" $\xi(x)$ remains as the task for a potential solver in this programming. For instance, the *SOSTools*© toolbox (PRAJNA *et al.*, 2004) manual suggests in Figure 16 that this vector is iteratively changed until a proper one is found. Mathematically, it means that the optimization problem faced by the solver is to minimize an objective function while assuring that $\tilde{H}(\xi(x), y) \triangleq \xi(x)\tilde{Q}(y)\xi(x)$ is a sum of squares, which is formally stated in (130).

$$H_{\text{SOS}}(x) = \xi'(x)F'F\xi(x) = \xi(x)\tilde{Q}(y)\xi(x) \Leftrightarrow H_{\text{SOS}}(x) \text{ is SOS} \tag{128}$$

$$\tilde{Q}(y) \triangleq \sum_{i=1}^{g} y_{(i)} Q_i \tag{129}$$

Figure 16 – Diagram depicting the solution stages of *SOSTools*©.



Source: PRAJNA *et al.*, 2004

$$\min_{y \in \mathbb{R}^g} cy$$
$$\text{subject to:} \tag{130}$$
$$\tilde{H}(\xi(x), y) \text{ is SOS}$$

7.2.1 On the issue of choosing the programming

In some chapters of this study, we change the formulation of the constraints from SDP to SOSP, and justify it with the reason that it allows a greater variety of LF candidates. Moreover, there are times when we adopt strategies to avoid a degeneration of the LMI formulation. In none of these situations an in-depth explanation is given, as they are considered intuitive for the reader that is familiar with LMIs.

Regardless, in this section of the Appendix, it is attempted to define the issue of changing programmings. The following propositions define the constraints of each programming:

Proposition 2. *Consider a matrix $L \in \mathbb{S}^n$. If the following set is not empty,*

$$\{y \in \mathbb{R}^g,\ \Omega_i,\ i = 1, ..., g \in \mathbb{S}^n\ :\ L = \sum_{i=1}^{g} y_{(i)} \Omega_i\} \tag{131}$$

then, $L \geq 0$ is a LMI.

Proposition 3. *Consider the UMP $H \in \Sigma^{h \times h}$. If the following set is not empty,*

$$\{\xi \in \Sigma^{\sigma \times h}, y \in \mathbb{R}^g,\ \Omega_i,\ i = 1, ..., g \in \mathbb{S}^n\ :\ H(x) = \xi(x) L \xi(x),\ L = \sum_{i=1}^{g} y_{(i)} \Omega_i\} \tag{132}$$

where $\Sigma^{\alpha_1, \alpha_2} = \{\{F : \mathbb{R} \to \mathbb{R}^{\alpha_1, \alpha_2}\} : F(x) = \sum_{i=0}^{d} F_i x^i,\ F_i \in \mathbb{R}^{\alpha_1, \alpha_2}\}\}$, then the following statement is valid:

$$L \geq 0 \Leftrightarrow H(x) \text{ is SOS} \Rightarrow H(x) \geq 0 \quad \forall x \tag{133}$$

To expose this issue, the Equation (54) from Chapter 4 is recalled:

$$P(\tau) = \sum_{i=0}^{d} P_i \tau^i = P_0 + \tau P_1 + \cdots + \tau^d P_d \tag{134}$$

This equation states that $P \in \Sigma^{n,n}$. Based on the propositions that have just been declared, it is understandable why $d > 1$ is problematic for the LMI formulation: taking into account that in this case the convexity argument is no longer valid (because $P(\tau)$ is not affine), τ cannot be fixed to a constant. Thus, τ must be an element of the vector of decision variables y. But there is no way to represent (134) as $y_{(1)}\Omega_1 + ... + y_{(g)}\Omega_g$, and consequently, considering $L = P$, the set given by (131) is empty (the proof of this statement is omitted). Hence, by Proposition 2, $P(\tau)$ cannot be in a LMI. Since the set (132) is not empty when $H = P$, it is valid to o ensure the nonnegativity of P with the SOS constraint in (133).

7.3 Schur complement of a symmetric matrix

The Schur complement is used to convert LMIs into nonlinear inequalities and vice-versa, and is given by the following lemma:

Lemma 5. (Schur complement) *The following condition*

$$\begin{bmatrix} M_1 & M_2 \\ \star & M_3 \end{bmatrix} \geq 0 \tag{135}$$

is equivalent to

$$M_1 - M_2 M_3^{-1} M_2 \geq 0$$

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