The Probability of Occurrence of Heywood Cases

Jandyra M.G. Fachel

Cadernos de Matemática e Estatística
Série A, n° 11, JAN/90
Porto Alegre, janeiro de 1990
The probability of occurrence of Heywood cases

A simulation study with the purpose of assessing the main causes of Heywood cases was performed (see Fachel, 1980). The results show that Heywood cases occur even when a factor model with large residual variances generated the data. In this paper we consider the case of one-factor models and we show how the probability of occurrence may be calculated. We also show that this probability for the one-factor model depends on sample size, parameter vectors and number of variables in the model. We shall consider the MINRES method of fitting for factor analysis, that is, we shall suppose that the factor loadings are estimated under the condition that the sum of squares of the off-diagonal residuals is minimized.

The MINRES method was introduced by Harman and Jones (1966) although, as they point out, the idea of getting a factor solution by minimizing off-residual correlations was first posed by Thurstone (1954). They do not consider the minimization of the total residual matrix (including diagonal terms), which would lead to the principal-factor solution. We do not consider the numerical procedures for obtaining the MINRES solution. It is on the principle of the method that we shall base our method for obtaining the probabilities of occurrence of Heywood cases.

Consider the one-factor model given by

\[ x_i = \lambda_i f + e_i \quad i=1,2,...,p \quad (1.1) \]

with \( \text{E}(f) = \text{E}(e_i) = 0 \); \( \text{E}(fe_i) = 0 \). Suppose \( \text{Var}(f) = 1 \), \( \text{E}(e_i) = 0 \), \( i \neq j \) and \( \text{Var}(e_i) = \psi \geq 0 \), \( i=1,...,p \). Suppose the \( x_i \)'s are standardized so that \( \text{Var}(x_i) = 1 \). Hence,

\[ \lambda_i^2 + \psi = 1, \quad i = 1,2,...,p \quad (1.2) \]

Then, we have

\[ \lambda_i^2 \leq 1, \quad i = 1,2,...,p \quad (1.3) \]

For this model

\[ \text{corr}(x_i, x_j) = \rho_{ij} = \lambda_i \lambda_j \quad (1.4) \]

Suppose, therefore, we fit the model by minimizing

\[ S = \sum_{i=1}^{p} \sum_{j=i+1}^{p} (r_{ij} - \lambda_i \lambda_j)^2 \quad (1.5) \]
where \( r_{ij} \) is the observed correlation between variables \( x_i \) and \( x_j \). (Actually, as \( x_i \) and \( x_j \) are standardized variables, \( r_{ij} \) and \( \rho_{ij} \) may be considered as covariances).

From a purely mathematical point of view, the problem of finding a minimum for the nonlinear function \( S \) is well defined. \( S \) is a function of the \( p(p-1)/2 \) off-diagonal residual correlations, which are dependent upon the elements of the factor matrix \( \Lambda(p \times 1) \). The minimum value for \( S \) occurs at the point where its partial derivatives with respect to the \( p \) elements of \( \Lambda \) are zero and its matrix of second derivatives is positive definite. Thus,

\[
S'(\lambda_i) = \frac{1}{2} \frac{\partial^2 S}{\partial \lambda_i^2} = \sum_{j \neq i} \lambda_{ij} + \lambda_i \sum_{j=1}^{p+1} \lambda_j^2 \quad i=1,2,\ldots,p
\]

we also have,

\[
\frac{1}{2} \frac{\partial^2 S}{\partial \lambda_i \partial \lambda_j} = \sum_{j \neq i} \lambda_j \geq 0
\]

\[
\frac{1}{2} \frac{\partial^2 S}{\partial \lambda_i \partial \lambda_j} = -r_{ij} + 2\lambda_i \lambda_j
\]

If the system of equations

\[
\frac{\partial S}{\partial \lambda_i} = 0 \quad i=1,2,\ldots,p
\]

has a solution with \( h_i^2 \leq 1 \), then it is clearly a minimum by (1.7). However, there may not be such a solution. For fixed \( \lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_p \), \( S'(\lambda_i) \) is a linear increasing function of \( \lambda_i \). We are interested in the behaviour of the function at the point \( \lambda_i = 1 \) and consequently \( y_i = 0 \). If the minimum of the function occurs at \( \lambda_i = 1 \), an exact Heywood case occurs. Let us consider the following cases:

1) Suppose that the derivative of \( S \) with respect to \( \lambda_i \) at the point \( \lambda_i = 1 \) is negative, that is \( S'(1) < 0 \).
At \( \lambda_i = 1 \), from (1.6) we have
\[
\left[ \frac{\partial S}{\partial \lambda_i} \right]_{\lambda_i = 1} = -\sum_{j=1}^{p} \lambda_j r_{ij} + \sum_{j \neq i}^{p} \lambda_j^2
\]

If \( S'(1) < 0 \), this would imply that \( S'(0) < 0 \) and that there will be no intermediate value of \( \lambda_i \) for which it is zero. The minimum of \( S \) will thus occur at \( \lambda_i = 1 \), because \( S' \) is a linear increasing function of \( \lambda_i \). A proof of this is given below:

Suppose \( S'(1) < 0 \) then

\[
-\sum_{j=1}^{p} \lambda_j r_{ij} + \sum_{j \neq i}^{p} \lambda_j^2 < 0
\]

\[
\sum_{j \neq i}^{p} \lambda_j r_{ij} > \sum_{j \neq i}^{p} \lambda_j^2 > 0
\]

Therefore

\[
S'(0) = \left[ \frac{\partial S}{\partial \lambda_i} \right]_{\lambda_i = 0} = -\sum_{j \neq i}^{p} \lambda_j r_{ij} < 0
\]

We recall that we are assuming \( 0 \leq \lambda_i \leq 1 \). It is known that for the one-factor model, \( A \) reduces to a column vector of \( p \) elements that is unique apart from a possible change of sign of all its elements, which corresponds merely to changing the sign of the factor. According to Lawley and Maxwell (1971), such changes are merely trivial.

2) Suppose now that \( S'(1) \) is positive. Therefore \( S'(0) \) may be negative or positive and the minimum of \( S \) will not occur at the point \( \lambda_i = 1 \).

3) Finally, if \( S'(1) = 0 \), the minimum of \( S \) will occur at \( \lambda_i = 1 \) and a Heywood case occur.

We are interested in evaluating the probability of an occurrence of a Heywood case for variable \( x_i \). Therefore we need consider only cases 1) and 3) above. That is
\[ \Pr \{ S'(1) \leq 0 \} = \Pr \left\{ \sum_{j=1 \atop j \neq i}^{p} \lambda_{r_{ij}} \geq \sum_{j=1}^{p} \lambda_{j}^{z} \right\} \quad i=1,2,\ldots,p \]  

(1.8)

For evaluating this probability we need to know the distribution of \( \sum_{j=1 \atop j \neq i}^{p} \lambda_{r_{ij}} \). An approximation may be obtained as follows.

Since the \( x \)'s have zero means and unit variances

\[ r_{ij} = \frac{1}{n} \sum_{h=1}^{n} x_{ih} x_{jh} \]

where \( h \) indexes the sample members. (The terms of this sum are independent for \( h=1,2,\ldots,n \); for sufficiently large \( n \), this will be approximately normal by the central limit theorem).

Consider

\[ z_{i} = \sum_{j=1 \atop j \neq i}^{p} \lambda_{r_{ij}} = \frac{1}{n} \sum_{h=1}^{n} x_{ih} \sum_{j=1 \atop j \neq i}^{p} \lambda_{j} \]

Let us denote \( \sum_{j=1 \atop j \neq i}^{p} \lambda_{j} \) by \( y_{ih} \) say. Therefore

\[ z_{i} = \frac{1}{n} \sum_{h=1}^{n} x_{ih} y_{ih} \]

We shall now find the moments of \( x_{ih} \) and \( y_{ih} \). By definition we have

\[ \mathbb{E}(x_{ih}) = 0, \quad \text{var}(x_{ih}) = 1 \]

Thus

\[ \mathbb{E}(y_{ih}) = 0 \]

\[ \text{var}(y_{ih}) = \text{var}(\sum_{j=1 \atop j \neq i}^{p} x_{jh} \lambda_{j}) = \sum_{j=1}^{p} \lambda_{j}^{2} + \sum_{j=1}^{p} \sum_{k=1}^{p} \lambda_{j} \lambda_{k} \quad \text{cov}(x_{jh},x_{kh}) \]

\[ j \neq i \quad j \neq k \]

(1.9)

Now
cov(x_{th}, x_{kh}) = \lambda \lambda_{jk} \tag{1.10}

thus
\begin{align*}
\text{var}(y_{th}) &= \sum_{j=1}^{p} \sum_{j=1}^{p} \sum_{j \neq k}^{p} \sum_{j \neq k}^{p} \lambda_{jk}^2 \\
&= \sum_{j=1}^{p} \sum_{j=1}^{p} \lambda_{jk}^2 \\
&= \lambda_{ij} \sum_{j=1}^{p} \lambda_{ik}^2
\end{align*} \tag{1.11}

\text{cov}(x_{th}, y_{th}) = \frac{p}{j=1} \sum_{j \neq i}^{p} \lambda_{jk}^2 \sum_{j \neq k}^{p} \lambda_{ik}^2 = \text{EC}(x_{th}, y_{th}) \tag{1.12}

Let us suppose that the observed variables \( x_i, i=1,\ldots,p \) have a sum of \( p-1 \) dependent normal variables, is also normal with mean and variance given by (1.9). Hence \( x_{th} \) and \( y_{th} \) have a normal bivariate distribution. From Kendall and Stuart (1977, Vol. I, p.85) it is known that if \( (x_{th}, y_{th}) \) are normal bivariate we have, using the fact that \( \sigma^2 = 1 \)

\[ \text{EC}(x_{th}, y_{th}) = (1 + 2 \rho_{x_{th} y_{th}}) \sigma_{x_{th}}^2 \sigma_{y_{th}}^2 = \sigma_{y_{th}}^2 + 2 \left[ \text{EC}(x_{th}, y_{th}) \right]^2 \]

Thus
\[ \text{Var}(x_{th} y_{th}) = \text{EC}(x_{th} y_{th})^2 - \left[ \text{EC}(x_{th} y_{th}) \right]^2 = \sigma_{y_{th}}^2 + 2 \left[ \text{EC}(x_{th} y_{th}) \right]^2 - \left[ \text{EC}(x_{th} y_{th}) \right]^2 = \sigma_{y_{th}}^2 + \left[ \text{EC}(x_{th} y_{th}) \right]^2 \tag{1.13} \]

Now
\[ z_i = \frac{1}{n} \sum_{h=1}^{n} x_{th} y_{th} \]

where \( z_i = x_{th} y_{th} \) are independent for \( h=1,\ldots,n \). Therefore, using (1.12)
\[ \text{EC}(z_i) = \frac{n}{n} \text{EC}(x_{th} y_{th}) = \lambda \sum_{j=1}^{p} \lambda_{jk}^2 \tag{1.14} \]
and using (1.11) and (1.13) we have

\[ \text{Var}(z_i) = \frac{1}{n} \left[ \sum_{j=1}^{p} \lambda_i^2 + \sum_{j=1}^{p} \sum_{k=1}^{p} \lambda_i^2 \lambda_j^2 + \lambda_i^2 \left( \sum_{j=1}^{p} \lambda_j^2 \right)^2 \right] \]  
(1.15)

For \( n \) sufficiently large

\[ z_i = \frac{1}{n} \sum_{h=1}^{n} X_{ih} Y_{ih} = \sum_{j \neq i} \lambda_i \lambda_j \]

is approximately normal, by the central limit theorem, with mean and variance given by (1.14) and (1.15) respectively. Therefore (1.8) may be calculated approximately as

\[ 1 - \phi \left[ \frac{\sum_{j \neq i} \lambda_i^2 - \lambda_i^2 \sum_{j \neq i} \lambda_j^2}{\frac{1}{\sqrt{n}} \left[ \sum_{j \neq i} \lambda_i^2 + \sum_{j \neq i} \lambda_i^2 \left( \sum_{j \neq i} \lambda_j^2 \right)^2 + \sum_{j \neq i} \sum_{k \neq i} \lambda_j^2 \lambda_k^2 \right]^{1/2}} \right] \]

\[ i=1,2,\ldots,p \]  
(1.16)

where \( \phi(z) \) is the normal distribution function.

The above expression gives the asymptotic probability of occurrence of a Heywood case for variable \( x_i \) in function of the sample size, magnitude of the factor loading parameters and the number \( p \) of variables in the model. As can be seen, if \( n \to \infty \), the expression (1.16) converges to zero. If \( \lambda_i \to 1 \), the probability of Heywood case converges to 0.5 for any values of \( \lambda_j (j \neq i) \).

We now present tables of the probability of occurrence of a Heywood case for several values of sample size and different values of \( p \), considering

1) \( \lambda_i = 0.98 \) and \( \lambda_j = 0.5 \) (\( j \neq i \))
2) \( \lambda_i = 0.90 \) and \( \lambda_j = 0.5 \) (\( j \neq i \)) \( i,j=1,2,\ldots,p \)
3) \( \lambda_i = 0.80 \) and \( \lambda_j = 0.5 \) (\( j \neq i \))
4) \( \lambda_i = \lambda_j = 0.50 \) \( i,j=1,2,\ldots,p \)
<table>
<thead>
<tr>
<th>( \lambda_i = \lambda_j = 0.90 )</th>
<th>i, j = 1, 2, ..., p</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( \lambda_i = 0.90 )</td>
</tr>
</tbody>
</table>

### Table 1.10 - Asymptotic probability of occurrence of Heywood cases for various sample sizes (n), different number of variables (p) and different magnitude of the factor loading parameters (cases 1 to 5)

#### 1) \( \lambda_i = 0.90 \), \( \lambda_j = 0.5 \) (i \( \neq \) j), i, j = 1, 2, ..., p

<table>
<thead>
<tr>
<th>n</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.4655</td>
<td>.4518</td>
<td>.4318</td>
<td>.4040</td>
<td>.3929</td>
<td>.3504</td>
<td>.2935</td>
<td>.1952</td>
</tr>
<tr>
<td>10</td>
<td>.4628</td>
<td>.4474</td>
<td>.4256</td>
<td>.3953</td>
<td>.3638</td>
<td>.3360</td>
<td>.2772</td>
<td>.1750</td>
</tr>
<tr>
<td>15</td>
<td>.4618</td>
<td>.4461</td>
<td>.4240</td>
<td>.3932</td>
<td>.3606</td>
<td>.3341</td>
<td>.2722</td>
<td>.1699</td>
</tr>
<tr>
<td>20</td>
<td>.4613</td>
<td>.4453</td>
<td>.4230</td>
<td>.3917</td>
<td>.3703</td>
<td>.3320</td>
<td>.2695</td>
<td>.1656</td>
</tr>
<tr>
<td>30</td>
<td>.4508</td>
<td>.4447</td>
<td>.4220</td>
<td>.3904</td>
<td>.3679</td>
<td>.3300</td>
<td>.2670</td>
<td>.1627</td>
</tr>
<tr>
<td>40</td>
<td>.4505</td>
<td>.4443</td>
<td>.4215</td>
<td>.3897</td>
<td>.3670</td>
<td>.3289</td>
<td>.2655</td>
<td>.1610</td>
</tr>
<tr>
<td>50</td>
<td>.4504</td>
<td>.4441</td>
<td>.4211</td>
<td>.3892</td>
<td>.3765</td>
<td>.3283</td>
<td>.2647</td>
<td>.1600</td>
</tr>
<tr>
<td>100</td>
<td>.4601</td>
<td>.4436</td>
<td>.4207</td>
<td>.3887</td>
<td>.3759</td>
<td>.3274</td>
<td>.2635</td>
<td>.1589</td>
</tr>
</tbody>
</table>

#### 2) \( \lambda_i = 0.90 \), \( \lambda_j = 0.5 \) (i \( \neq \) j), i, j = 1, 2, ..., p

<table>
<thead>
<tr>
<th>n</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.3293</td>
<td>.2860</td>
<td>.1884</td>
<td>.1057</td>
<td>.0812</td>
<td>.0241</td>
<td>.0026</td>
<td>.0000</td>
</tr>
<tr>
<td>10</td>
<td>.3148</td>
<td>.2473</td>
<td>.1471</td>
<td>.0660</td>
<td>.0634</td>
<td>.0154</td>
<td>.0012</td>
<td>.0000</td>
</tr>
<tr>
<td>15</td>
<td>.3097</td>
<td>.2411</td>
<td>.1502</td>
<td>.0600</td>
<td>.0580</td>
<td>.0131</td>
<td>.0009</td>
<td>.0000</td>
</tr>
<tr>
<td>20</td>
<td>.3070</td>
<td>.2380</td>
<td>.1557</td>
<td>.0770</td>
<td>.0555</td>
<td>.0121</td>
<td>.0008</td>
<td>.0000</td>
</tr>
<tr>
<td>30</td>
<td>.3047</td>
<td>.2349</td>
<td>.1534</td>
<td>.0740</td>
<td>.0531</td>
<td>.0112</td>
<td>.0007</td>
<td>.0000</td>
</tr>
<tr>
<td>40</td>
<td>.3034</td>
<td>.2334</td>
<td>.1517</td>
<td>.0727</td>
<td>.0518</td>
<td>.0107</td>
<td>.0006</td>
<td>.0000</td>
</tr>
<tr>
<td>50</td>
<td>.3026</td>
<td>.2324</td>
<td>.1508</td>
<td>.0719</td>
<td>.0511</td>
<td>.0104</td>
<td>.0005</td>
<td>.0000</td>
</tr>
<tr>
<td>100</td>
<td>.3012</td>
<td>.2306</td>
<td>.1487</td>
<td>.0703</td>
<td>.0497</td>
<td>.0099</td>
<td>.0005</td>
<td>.0000</td>
</tr>
</tbody>
</table>
3) \( \lambda_i = 0.80 \quad \lambda_j = 0.5 \quad (j \neq 1) \quad i, j = 1, 2, \ldots p \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.1802</td>
<td>.0680</td>
<td>.0339</td>
<td>.0048</td>
<td>.0019</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>10</td>
<td>.1572</td>
<td>.0773</td>
<td>.0220</td>
<td>.0022</td>
<td>.0007</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>15</td>
<td>.1495</td>
<td>.0710</td>
<td>.0189</td>
<td>.0016</td>
<td>.0005</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>20</td>
<td>.1457</td>
<td>.0679</td>
<td>.0174</td>
<td>.0014</td>
<td>.0004</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>30</td>
<td>.1421</td>
<td>.0650</td>
<td>.0161</td>
<td>.0013</td>
<td>.0003</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>40</td>
<td>.1405</td>
<td>.0635</td>
<td>.0155</td>
<td>.0011</td>
<td>.0003</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>50</td>
<td>.1392</td>
<td>.0626</td>
<td>.0151</td>
<td>.0011</td>
<td>.0003</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>100</td>
<td>.1370</td>
<td>.0609</td>
<td>.0143</td>
<td>.0010</td>
<td>.0003</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
</tbody>
</table>

4) \( \lambda_i = \lambda_j = 0.50 \quad i, j = 1, 2, \ldots p \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>.0062</td>
<td>.0002</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>10</td>
<td>.0025</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>15</td>
<td>.0017</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>20</td>
<td>.0015</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>30</td>
<td>.0012</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>40</td>
<td>.0011</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>50</td>
<td>.0010</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
<tr>
<td>100</td>
<td>.0009</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
<td>.0000</td>
</tr>
</tbody>
</table>
5) \( \lambda_i = \lambda_j = 0.90 \), \( i, j = 1, 2, \ldots, p \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>50</th>
<th>100</th>
<th>200</th>
<th>400</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>5000</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.3025</td>
<td>0.2322</td>
<td>0.1507</td>
<td>0.0717</td>
<td>0.0510</td>
<td>0.0103</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>10</td>
<td>0.3009</td>
<td>0.2303</td>
<td>0.1483</td>
<td>0.0700</td>
<td>0.0494</td>
<td>0.0099</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>15</td>
<td>0.3004</td>
<td>0.2297</td>
<td>0.1477</td>
<td>0.0694</td>
<td>0.0490</td>
<td>0.0097</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>20</td>
<td>0.3002</td>
<td>0.2294</td>
<td>0.1474</td>
<td>0.0692</td>
<td>0.0488</td>
<td>0.0096</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>30</td>
<td>0.3000</td>
<td>0.2292</td>
<td>0.1471</td>
<td>0.0690</td>
<td>0.0486</td>
<td>0.0095</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>40</td>
<td>0.2999</td>
<td>0.2290</td>
<td>0.1457</td>
<td>0.0689</td>
<td>0.0485</td>
<td>0.0095</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>50</td>
<td>0.2998</td>
<td>0.2288</td>
<td>0.1457</td>
<td>0.0688</td>
<td>0.0484</td>
<td>0.0094</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
<tr>
<td>100</td>
<td>0.2997</td>
<td>0.2286</td>
<td>0.1457</td>
<td>0.0687</td>
<td>0.0484</td>
<td>0.0094</td>
<td>0.0005</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
Final comments

The tables of probability of Heywood case are obtained from considerations about the principle of the minres method, that is, minimizing the sum of squares of the off-diagonal residual correlations. We also believe that some of the constraints in the algorithm for the MLFA method, of the main statistical packages, as for example stopping the iteration for any particular variable which \( y_i = 0 \), could lead to a different proportion of Heywood cases in some situations, because of a possible bias in the method.

For large sample sizes, the probability of Heywood cases is very small. We observe in Table 1.10 that the probability of Heywood cases decreases as 1) the sample size increases; 2) the number of variables in the model increases but a small variation is observed from \( p=5 \) to \( p=100 \) for all tables. Related to the magnitude of the factor loading parameters, we observe that when only one factor loading increases (the others being equal to 0.5), the probability of Heywood cases also increases. Finally, it is interesting to note, comparing cases 2\( \delta \) and 5\( \delta \) in Table 1.10, that when only one loading is equal to 0.90, the probabilities of Heywood cases are slightly higher than when all loadings are equal to 0.90.

The approximate probability of the occurrence of Heywood cases for different parameter vectors for the one-factor model may be easily evaluated using expression (1.18). In Table 1.10 we have included only some cases.
REFERENCES


Série A: Trabalho de Pesquisa

3. Eduardo Cisneros, Miguel Ferrero e Maria Inés Gonzales - Prime Ideals of Skew Polynomial Rings and Skew Laurent Polynomial Rings - ABR/89.
4. Oclide José Dotto - $\varepsilon$ - Dilations - JUN/89.
8. Jaime Bruck Ripoll - A Note on Compact Surfaces with Non Zero Constant Mean Curvature - OUT/89.
Instituto de Matemática
Diretor: Professor Aron Taitelbaum
Núcleo de Atividades Extra Curriculares
Coordenador: Professora Jandyra G. Fachel
Secretária: Rosaura Monteiro Pinheiro

Os Cadernos de Matemática e Estatística publicam as seguintes séries:
Série A: Trabalho de Pesquisa
Série B: Trabalho de Apoio Didático
Série C: Colóquio de Matemática SBM/UFRGS
Série D: Trabalho de Graduação
Série E: Dissertações de Mestrado
Série F: Trabalho de Divulgação
Série G: Textos para Discussão

Toda correspondência com solicitação de números publicados e demais informações
deverá ser enviada para:
NAEC - Núcleo de Atividades Extra Curriculares
Instituto de Matemática - UFRGS
Av. Bento Gonçalves, 9500
91.500 - Agronomía - POA/RS
Telefone: 36.11.59 ou 36.17.85 Ramal: 252